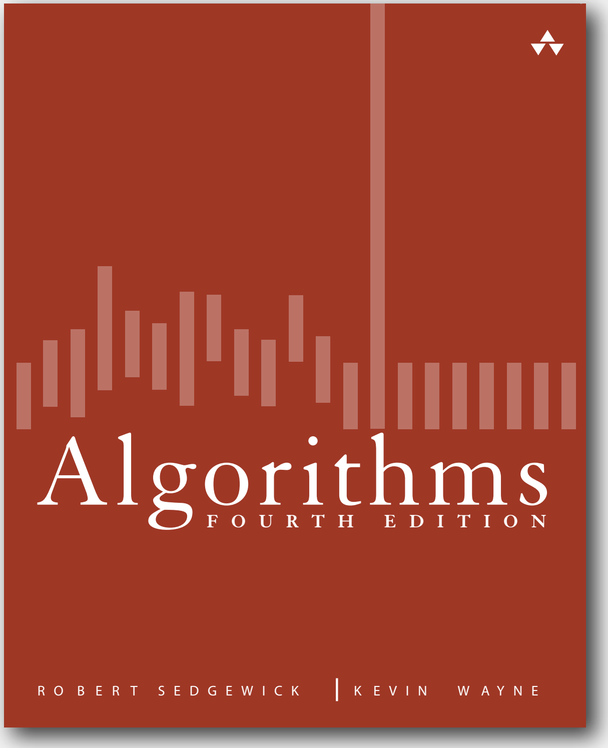
# 5. STRINGS



# String processing

String. Sequence of characters.

Important fundamental abstraction.

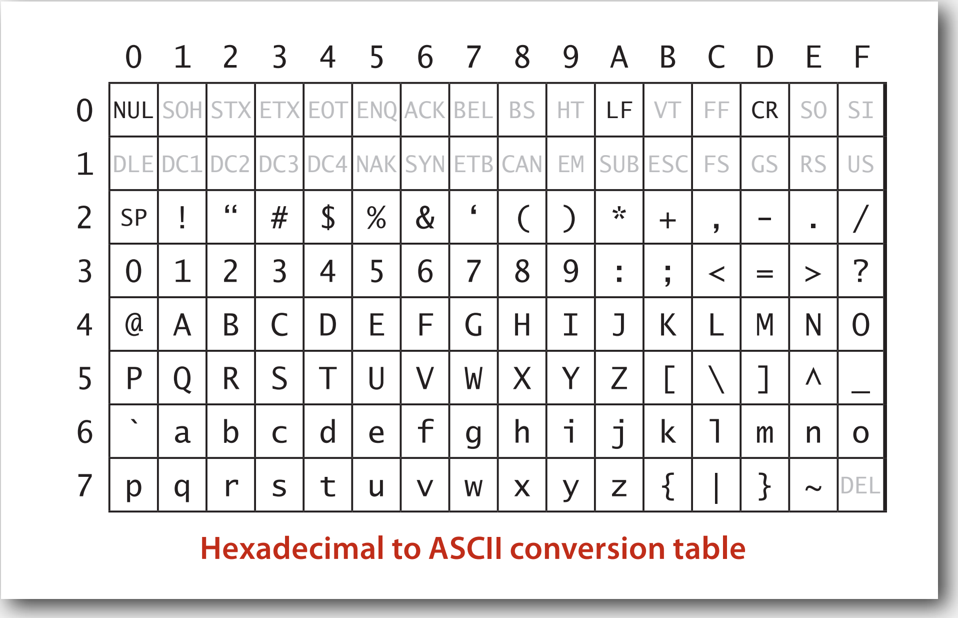
- Information processing.
- Genomic sequences.
- Communication systems (e.g., email).
- Programming systems (e.g., Java programs).
- …

> *" The digital information that underlies biochemistry, cell biology, and development can be represented by a simple string of G's, A's, T's and C's. This string is the root data structure of an organism's biology. "* — *M. V. Olson*

# The char data type

C char data type. Typically an 8-bit integer.

- Supports 7-bit ASCII.
- Need more bits to represent certain characters.



| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | A | B | C | D | E | F |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | NUL | SOH | STX | ETX | EOT | ENQ | ACK | BEL | BS | HT | LF | VT | FF | CR | SO | SI |
| 1 | DLE | DC1 | DC2 | DC3 | DC4 | NAK | SYN | ETB | CAN | EM | SUB | ESC | FS | GS | RS | US |
| 2 | SP | ! | “ | # | $ | % | & | ‘ | ( | ) | * | + | , | - | . | / |
| 3 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | : | ; | < | = | > | ? |
| 4 | @ | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
| 5 | P | Q | R | S | T | U | V | W | X | Y | Z | [ | \ | ] | ^ | _ |
| 6 | ` | a | b | c | d | e | f | g | h | i | j | k | l | m | n | o |
| 7 | p | q | r | s | t | u | v | w | x | y | z | { | \| | } | ~ | DEL |

Hexadecimal to ASCII conversion table

Java char data type. A 16-bit unsigned integer.

- Supports original 16-bit Unicode.
- Supports 21-bit Unicode 3.0 (awkwardly).

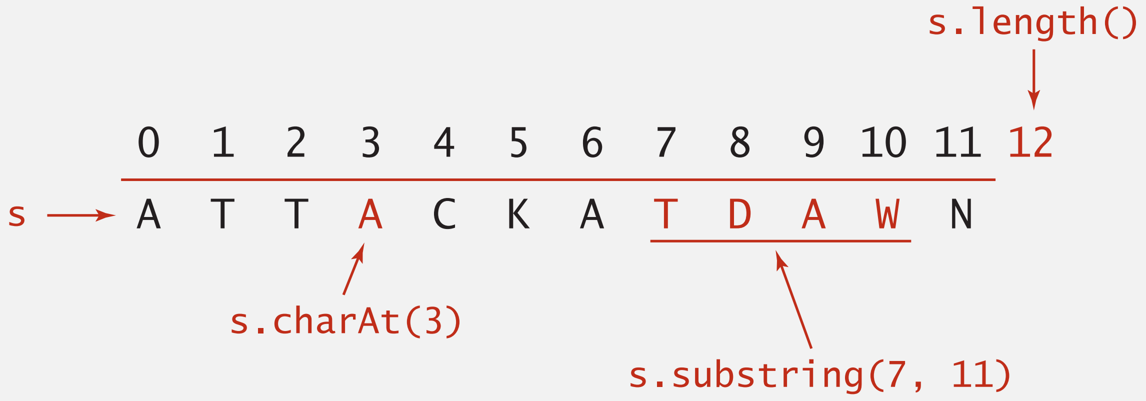
# The String data type

**String data type.** Sequence of characters (immutable).

**Indexing.** Get the $i^{th}$ character.

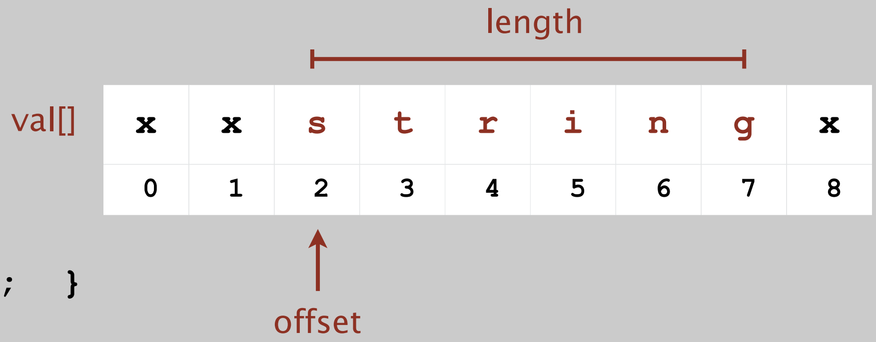
**Substring extraction.** Get a contiguous sequence of characters from a string.

**String concatenation.** Append one character to end of another string.



# The String data type: Java implementation

```
public final class String implements Comparable<String>
{
   private char[] val;    // characters
   private int offset;    // index of first char in array
   private int length;    // length of string
   private int hash;      // cache of hashCode()


   public int length()
   {  return length; }

   public char charAt(int i)
   {  return value[i + offset];  }


   private String(int offset, int length, char[] val)
   {
      this.offset = offset;
      this.length = length;
      this.val    = val;
   }

   public String substring(int from, int to)
   {  return new String(offset + from, to - from, val);  }
   …
```



copy of reference to original char array

# The String data type: performance

**String data type.** Sequence of characters (immutable).

**Underlying implementation.** Immutable `char[]` array, offset, and length.

|  | **`String`** |  |
|---|---|---|
| **operation** | **guarantee** | **extra space** |
| `charAt()` | 1 | 1 |
| `length()` | 1 | 1 |
| `substring()` | 1 | 1 |
| `concat()` | N | N |

**Memory.** $40 + 2N$ bytes for a virgin `String` of length $N$.

can use `byte[]` or `char[]` instead of `String` to save space
(but lose convenience of `String` data type)

# The StringBuilder data type

**StringBuilder data type.** Sequence of characters (mutable).

**Underlying implementation.** Resizing `char[]` array and length.

|  | `String` |  | `StringBuilder` |  |
|---|---|---|---|---|
| operation | guarantee | extra space | guarantee | extra space |
| `charAt()` | 1 | 1 | 1 | 1 |
| `length()` | 1 | 1 | 1 | 1 |
| `substring()` | 1 | 1 | N | N |
| `concat()` | N | N | 1 * | 1 * |

* amortized

**Remark.** `StringBuffer` data type is similar, but thread safe (and slower).

# String vs. StringBuilder

**Q.** How to efficiently reverse a string?

**A.**

```
public static String reverse(String s)
{
   String rev = "";
   for (int i = s.length() - 1; i >= 0; i--)
      rev += s.charAt(i);
   return rev;
}
```

← quadratic time

**B.**

```
public static String reverse(String s)
{
   StringBuilder rev = new StringBuilder();
   for (int i = s.length() - 1; i >= 0; i--)
      rev.append(s.charAt(i));
   return rev.toString();
}
```

← linear time

# String challenge: array of suffixes

Q. How to efficiently form array of suffixes?

**input string**

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | a | c | a | a | g | t | t | t | a | c | a | a | g | c |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

**suffixes**

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | a | a | c | a | a | g | t | t | t | a | c | a | a | g | c |
| 1 | a | c | a | a | g | t | t | t | a | c | a | a | g | c | |
| 2 | c | a | a | g | t | t | t | a | c | a | a | g | c | | |
| 3 | a | a | g | t | t | t | a | c | a | a | g | c | | | |
| 4 | a | g | t | t | t | a | c | a | a | g | c | | | | |
| 5 | g | t | t | t | a | c | a | a | g | c | | | | | |
| 6 | t | t | t | a | c | a | a | g | c | | | | | | |
| 7 | t | t | a | c | a | a | g | c | | | | | | | |
| 8 | t | a | c | a | a | g | c | | | | | | | | |
| 9 | a | c | a | a | g | c | | | | | | | | | |
| 10 | c | a | a | g | c | | | | | | | | | | |
| 11 | a | a | g | c | | | | | | | | | | | |
| 12 | a | g | c | | | | | | | | | | | | |
| 13 | g | c | | | | | | | | | | | | | |
| 14 | c | | | | | | | | | | | | | | |

# String vs. StringBuilder

**Q.** How to efficiently form array of suffixes?

**A.**

```
public static String[] suffixes(String s)
{
   int N = s.length();
   String[] suffixes = new String[N];
   for (int i = 0; i < N; i++)
      suffixes[i] = s.substring(i, N);
   return suffixes;
}
```

linear time and
linear space

**B.**

```
 public static String[] suffixes(String s)
 {
    int N = s.length();
    StringBuilder sb = new StringBuilder(s);
    String[] suffixes = new String[N];
    for (int i = 0; i < N; i++)
       suffixes[i] = sb.substring(i, N);
    return suffixes;
}
```

quadratic time and
quadratic space

# Longest common prefix

Q. How long to compute length of longest common prefix?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| p | r | e | f | e | t | c | h |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| p | r | e | f | i | x | | |

```
public static int lcp(String s, String t)
{
   int N = Math.min(s.length(), t.length());
   for (int i = 0; i < N; i++)
      if (s.charAt(i) != t.charAt(i))
         return i;
   return N;
}
```

linear time (worst case)
sublinear time (typical case)

**Running time.** Proportional to length $D$ of longest common prefix.

**Remark.** Also can compute `compareTo()` in sublinear time.

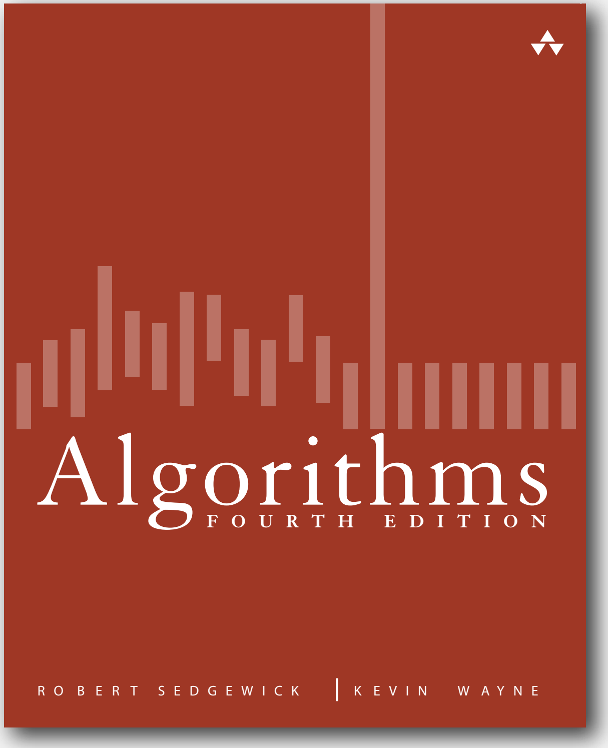
# Alphabets

**Digital key.** Sequence of digits over fixed alphabet.

**Radix.** Number of digits $R$ in alphabet.

| name | R() | lgR() | characters |
|---|---|---|---|
| BINARY | 2 | 1 | 01 |
| OCTAL | 8 | 3 | 01234567 |
| DECIMAL | 10 | 4 | 0123456789 |
| HEXADECIMAL | 16 | 4 | 0123456789ABCDEF |
| DNA | 4 | 2 | ACTG |
| LOWERCASE | 26 | 5 | abcdefghijklmnopqrstuvwxyz |
| UPPERCASE | 26 | 5 | ABCDEFGHIJKLMNOPQRSTUVWXYZ |
| PROTEIN | 20 | 5 | ACDEFGHIKLMNPQRSTVWY |
| BASE64 | 64 | 6 | ABCDEFGHIJKLMNOPQRSTUVWXYZabcdef<br>ghijklmnopqrstuvwxyz0123456789+/ |
| ASCII | 128 | 7 | *ASCII characters* |
| EXTENDED_ASCII | 256 | 8 | *extended ASCII characters* |
| UNICODE16 | 65536 | 16 | *Unicode characters* |

# 5.1 STRING SORTS



- key-indexed counting
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- suffix arrays

# Review: summary of the performance of sorting algorithms

Frequency of operations = key compares.

| algorithm | guarantee | random | extra space | stable? | operations on keys |
|---|---|---|---|---|---|
| insertion sort | $N^2 / 2$ | $N^2 / 4$ | 1 | yes | `compareTo()` |
| mergesort | N lg N | N lg N | N | yes | `compareTo()` |
| quicksort | 1.39 N lg N * | 1.39 N lg N | c lg N | no | `compareTo()` |
| heapsort | 2 N lg N | 2 N lg N | 1 | no | `compareTo()` |

* probabilistic

**Lower bound.** $\sim N \lg N$ compares are required by any compare-based algorithm.

**Q.** Can we do better (despite the lower bound)?

**A.** Yes, if we don't depend on key compares.

- **key-indexed counting**
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- suffix arrays

# Key-indexed counting: assumptions about keys

**Assumption.** Keys are integers between $0$ and $R - 1$.

**Implication.** Can use key as an array index.

**Applications.**

- Sort string by first letter.
- Sort class roster by section.
- Sort phone numbers by area code.
- Subroutine in a sorting algorithm. [stay tuned]

**Remark.** Keys may have associated data $\Rightarrow$ can't just count up number of keys of each value.

| **input** | | **sorted result** | |
|---|---|---|---|
| *name* | *section* | *(by section)* | |
| Anderson | 2 | Harris | 1 |
| Brown | 3 | Martin | 1 |
| Davis | 3 | Moore | 1 |
| Garcia | 4 | Anderson | 2 |
| Harris | 1 | Martinez | 2 |
| Jackson | 3 | Miller | 2 |
| Johnson | 4 | Robinson | 2 |
| Jones | 3 | White | 2 |
| Martin | 1 | Brown | 3 |
| Martinez | 2 | Davis | 3 |
| Miller | 2 | Jackson | 3 |
| Moore | 1 | Jones | 3 |
| Robinson | 2 | Taylor | 3 |
| Smith | 4 | Williams | 3 |
| Taylor | 3 | Garcia | 4 |
| Thomas | 4 | Johnson | 4 |
| Thompson | 4 | Smith | 4 |
| White | 2 | Thomas | 4 |
| Williams | 3 | Thompson | 4 |
| Wilson | 4 | Wilson | 4 |

↑ *keys are small integers*

# Key-indexed counting demo

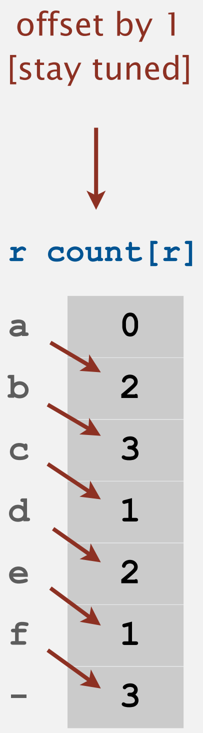
**Goal.** Sort an array `a[]` of $N$ integers between $0$ and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

count frequencies →

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

| i | a[i] |
|---|---|
| 0 | d |
| 1 | a |
| 2 | c |
| 3 | f |
| 4 | f |
| 5 | b |
| 6 | d |
| 7 | b |
| 8 | f |
| 9 | b |
| 10 | e |
| 11 | a |



offset by 1
[stay tuned]

| r | count[r] |
|---|---|
| a | 0 |
| b | 2 |
| c | 3 |
| d | 1 |
| e | 2 |
| f | 1 |
| - | 3 |

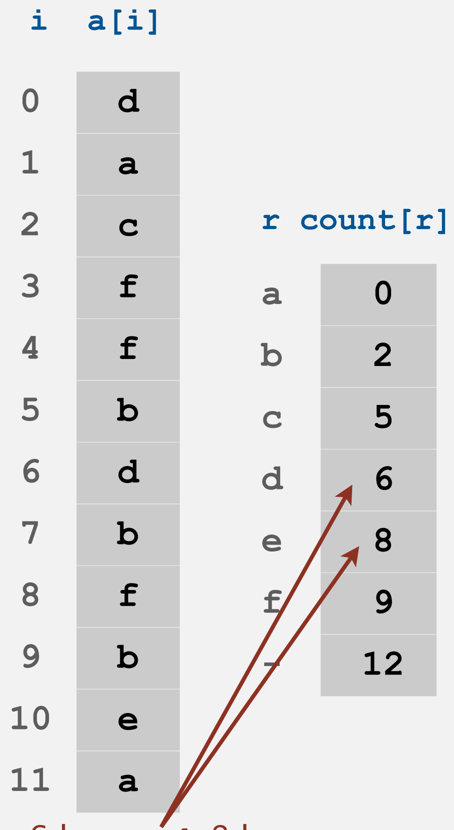
# Key-indexed counting demo

**Goal.** Sort an array `a[]` of $N$ integers between $0$ and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

compute cumulates →



| i | a[i] |
|---|---|
| 0 | d |
| 1 | a |
| 2 | c |
| 3 | f |
| 4 | f |
| 5 | b |
| 6 | d |
| 7 | b |
| 8 | f |
| 9 | b |
| 10 | e |
| 11 | a |

| r | count[r] |
|---|---|
| a | 0 |
| b | 2 |
| c | 5 |
| d | 6 |
| e | 8 |
| f | 9 |
| - | 12 |

6 keys < `d`, 8 keys < `e`
so `d`’s go in `a[6]` and `a[7]`

# Key-indexed counting demo

**Goal.** Sort an array `a[]` of $N$ integers between $0$ and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

move items →

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

| i | a[i] |
|---|---|
| 0 | d |
| 1 | a |
| 2 | c |
| 3 | f |
| 4 | f |
| 5 | b |
| 6 | d |
| 7 | b |
| 8 | f |
| 9 | b |
| 10 | e |
| 11 | a |

| r | count[r] |
|---|---|
| a | 2 |
| b | 5 |
| c | 6 |
| d | 8 |
| e | 9 |
| f | 12 |
| - | 12 |

| i | aux[i] |
|---|---|
| 0 | a |
| 1 | a |
| 2 | b |
| 3 | b |
| 4 | b |
| 5 | c |
| 6 | d |
| 7 | d |
| 8 | e |
| 9 | f |
| 10 | f |
| 11 | f |

# Key-indexed counting demo

**Goal.** Sort an array `a[]` of $N$ integers between $0$ and $R - 1$.

- Count frequencies of each letter using key as index.
- Compute frequency cumulates which specify destinations.
- Access cumulates using key as index to move items.
- Copy back into original array.

```
int N = a.length;
int[] count = new int[R+1];

for (int i = 0; i < N; i++)
   count[a[i]+1]++;

for (int r = 0; r < R; r++)
   count[r+1] += count[r];

for (int i = 0; i < N; i++)
   aux[count[a[i]]++] = a[i];

for (int i = 0; i < N; i++)
   a[i] = aux[i];
```

copy back

| i | a[i] |
|---|---|
| 0 | a |
| 1 | a |
| 2 | b |
| 3 | b |
| 4 | b |
| 5 | c |
| 6 | d |
| 7 | d |
| 8 | e |
| 9 | f |
| 10 | f |
| 11 | f |

| r | count[r] |
|---|---|
| a | 2 |
| b | 5 |
| c | 6 |
| d | 8 |
| e | 9 |
| f | 12 |
| - | 12 |

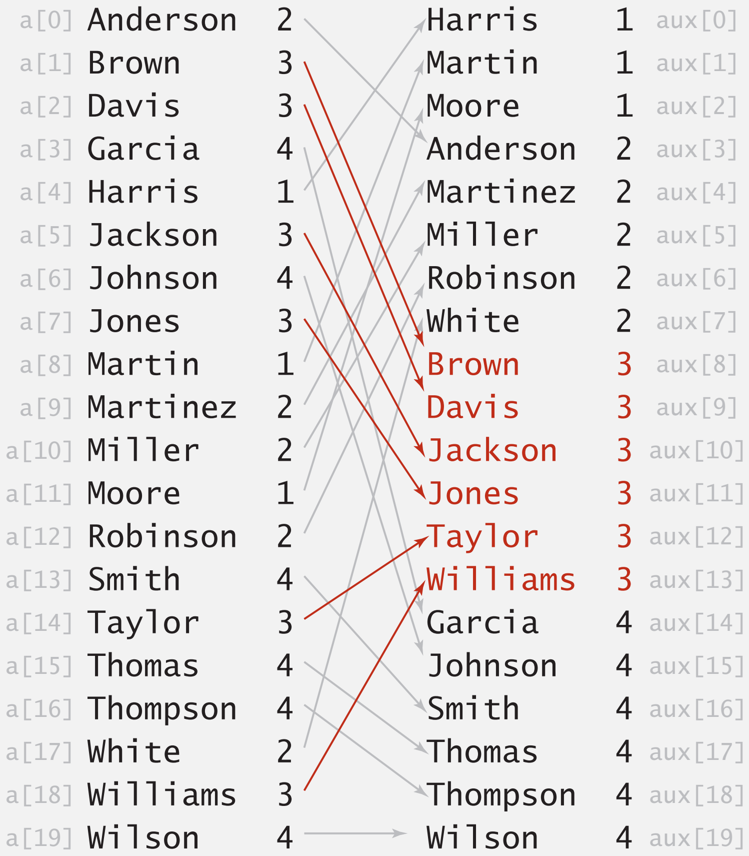
| i | aux[i] |
|---|---|
| 0 | a |
| 1 | a |
| 2 | b |
| 3 | b |
| 4 | b |
| 5 | c |
| 6 | d |
| 7 | d |
| 8 | e |
| 9 | f |
| 10 | f |
| 11 | f |

# Key-indexed counting: analysis

**Proposition.** Key-indexed counting uses $\sim 11\,N + 4\,R$ array accesses to sort $N$ items whose keys are integers between $0$ and $R - 1$.

**Proposition.** Key-indexed counting uses extra space proportional to $N + R$.

**Stable?** ✔



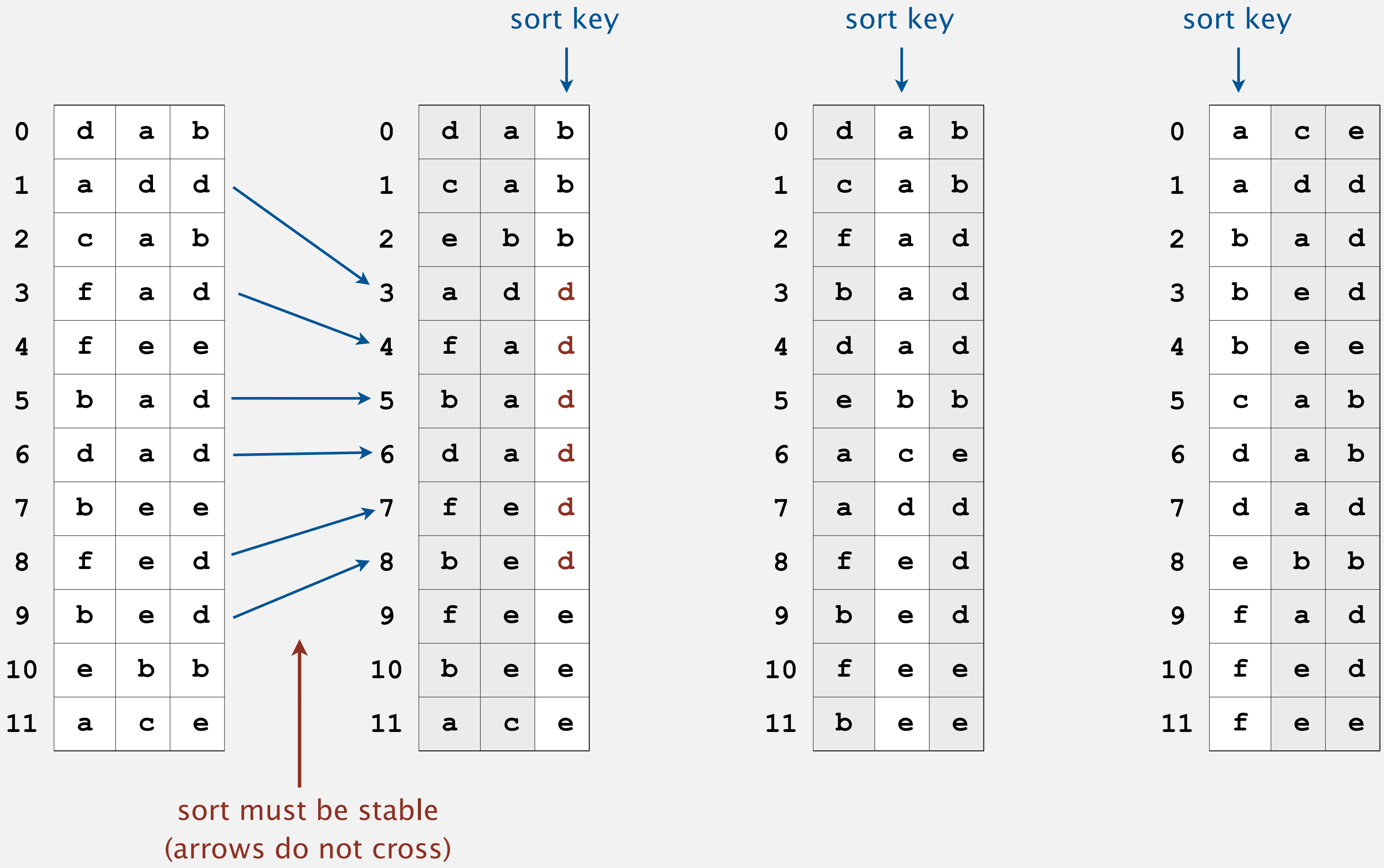
| | | | | | |
|---|---|---|---|---|---|
| a[0] | Anderson | 2 | Harris | 1 | aux[0] |
| a[1] | Brown | 3 | Martin | 1 | aux[1] |
| a[2] | Davis | 3 | Moore | 1 | aux[2] |
| a[3] | Garcia | 4 | Anderson | 2 | aux[3] |
| a[4] | Harris | 1 | Martinez | 2 | aux[4] |
| a[5] | Jackson | 3 | Miller | 2 | aux[5] |
| a[6] | Johnson | 4 | Robinson | 2 | aux[6] |
| a[7] | Jones | 3 | White | 2 | aux[7] |
| a[8] | Martin | 1 | Brown | 3 | aux[8] |
| a[9] | Martinez | 2 | Davis | 3 | aux[9] |
| a[10] | Miller | 2 | Jackson | 3 | aux[10] |
| a[11] | Moore | 1 | Jones | 3 | aux[11] |
| a[12] | Robinson | 2 | Taylor | 3 | aux[12] |
| a[13] | Smith | 4 | Williams | 3 | aux[13] |
| a[14] | Taylor | 3 | Garcia | 4 | aux[14] |
| a[15] | Thomas | 4 | Johnson | 4 | aux[15] |
| a[16] | Thompson | 4 | Smith | 4 | aux[16] |
| a[17] | White | 2 | Thomas | 4 | aux[17] |
| a[18] | Williams | 3 | Thompson | 4 | aux[18] |
| a[19] | Wilson | 4 | Wilson | 4 | aux[19] |

- key-indexed counting
- **LSD radix sort**
- MSD radix sort
- 3-way radix quicksort
- suffix arrays

# Least-significant-digit-first string sort

## LSD string (radix) sort.

- Consider characters from right to left.
- Stably sort using $d^{th}$ character as the key (using key-indexed counting).



sort key

sort key

sort key

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | d | a | b | 0 | d | a | b | 0 | d | a | b | 0 | a | c | e |
| 1 | a | d | d | 1 | c | a | b | 1 | c | a | b | 1 | a | d | d |
| 2 | c | a | b | 2 | e | b | b | 2 | f | a | d | 2 | b | a | d |
| 3 | f | a | d | 3 | a | d | d | 3 | b | a | d | 3 | b | e | d |
| 4 | f | e | e | 4 | f | a | d | 4 | d | a | d | 4 | b | e | e |
| 5 | b | a | d | 5 | b | a | d | 5 | e | b | b | 5 | c | a | b |
| 6 | d | a | d | 6 | d | a | d | 6 | a | c | e | 6 | d | a | b |
| 7 | b | e | e | 7 | f | e | d | 7 | a | d | d | 7 | d | a | d |
| 8 | f | e | d | 8 | b | e | d | 8 | f | e | d | 8 | e | b | b |
| 9 | b | e | d | 9 | f | e | e | 9 | b | e | d | 9 | f | a | d |
| 10 | e | b | b | 10 | b | e | e | 10 | f | e | e | 10 | f | e | d |
| 11 | a | c | e | 11 | a | c | e | 11 | b | e | e | 11 | f | e | e |

sort must be stable
(arrows do not cross)

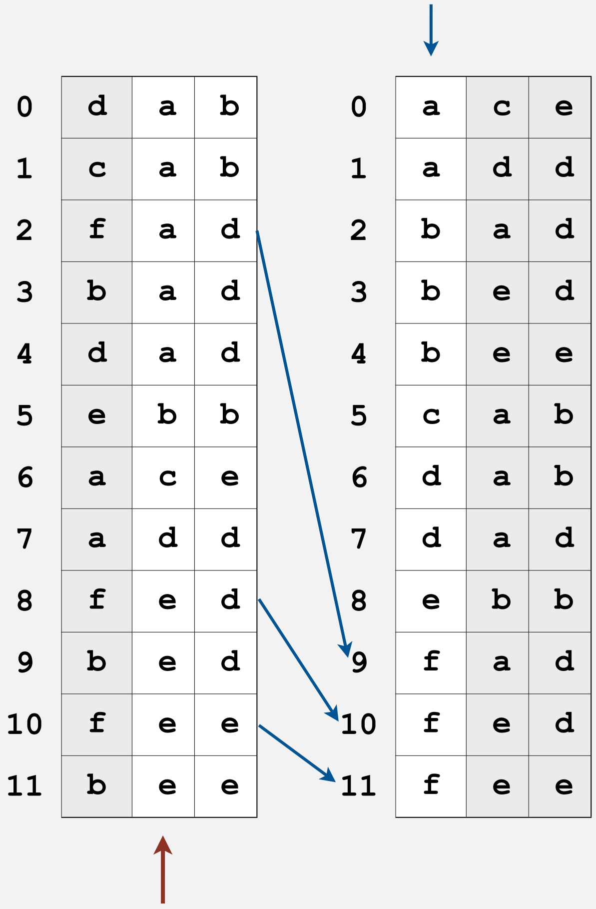
# LSD string sort: correctness proof

**Proposition.** LSD sorts fixed-length strings in ascending order.

**Pf.** [by induction]

**Invariant.** After pass $i$, the strings are sorted by last $i$ characters.

- If two strings differ on sort key, key-indexed sort puts them in proper relative order.
- If two strings agree on sort key, stability keeps them in proper relative order.



sort key

| | | | |
|---|---|---|---|
| 0 | d | a | b |
| 1 | c | a | b |
| 2 | f | a | d |
| 3 | b | a | d |
| 4 | d | a | d |
| 5 | e | b | b |
| 6 | a | c | e |
| 7 | a | d | d |
| 8 | f | e | d |
| 9 | b | e | d |
| 10 | f | e | e |
| 11 | b | e | e |

| | | | |
|---|---|---|---|
| 0 | a | c | e |
| 1 | a | d | d |
| 2 | b | a | d |
| 3 | b | e | d |
| 4 | b | e | e |
| 5 | c | a | b |
| 6 | d | a | b |
| 7 | d | a | d |
| 8 | e | b | b |
| 9 | f | a | d |
| 10 | f | e | d |
| 11 | f | e | e |

sorted from previous passes (by induction)

# LSD string sort: Java implementation

```
public class LSD
{
   public static void sort(String[] a, int W)
   {
      int R = 256;
      int N = a.length;
      String[] aux = new String[N];

      for (int d = W-1; d >= 0; d--)
      {
         int[] count = new int[R+1];
         for (int i = 0; i < N; i++)
            count[a[i].charAt(d) + 1]++;
         for (int r = 0; r < R; r++)
            count[r+1] += count[r];
         for (int i = 0; i < N; i++)
            aux[count[a[i].charAt(d)]++] = a[i];
         for (int i = 0; i < N; i++)
            a[i] = aux[i];
      }
   }
}
```

fixed-length W strings

radix R

do key-indexed counting
for each digit from right to left

key-indexed
counting

# Summary of the performance of sorting algorithms

Frequency of operations.

| algorithm | guarantee | random | extra space | stable? | operations on keys |
|---|---|---|---|---|---|
| insertion sort | $N^2 / 2$ | $N^2 / 4$ | 1 | yes | `compareTo()` |
| mergesort | N lg N | N lg N | N | yes | `compareTo()` |
| quicksort | 1.39 N lg N * | 1.39 N lg N | c lg N | no | `compareTo()` |
| heapsort | 2 N lg N | 2 N lg N | 1 | no | `compareTo()` |
| LSD † | 2 W N | 2 W N | N + R | yes | `charAt()` |

* probabilistic

† fixed-length W keys

**Q.** What if strings do not have same length?

# String sorting challenge 1

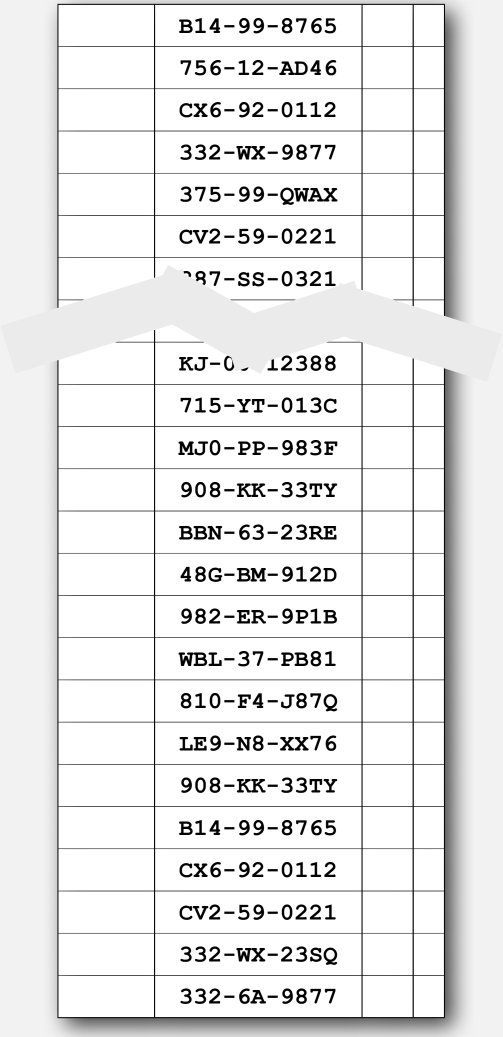
**Problem.** Sort a huge commercial database on a fixed-length key.

**Ex.** Account number, date, Social Security number, ...

## Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD string sort.

256 (or 65,536) counters;
Fixed-length strings sort in W passes.



| |
|---|
| B14-99-8765 |
| 756-12-AD46 |
| CX6-92-0112 |
| 332-WX-9877 |
| 375-99-QWAX |
| CV2-59-0221 |
| [illegible]87-SS-0321 |
| KJ-0[illegible]12388 |
| 715-YT-013C |
| MJ0-PP-983F |
| 908-KK-33TY |
| BBN-63-23RE |
| 48G-BM-912D |
| 982-ER-9P1B |
| WBL-37-PB81 |
| 810-F4-J87Q |
| LE9-N8-XX76 |
| 908-KK-33TY |
| B14-99-8765 |
| CX6-92-0112 |
| CV2-59-0221 |
| 332-WX-23SQ |
| 332-6A-9877 |

# String sorting challenge 2a

Problem. Sort one million 32-bit integers.

Ex. Google (or presidential) interview.

Which sorting method to use?

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD string sort.

**Google CEO Eric Schmidt interviews Barack Obama**

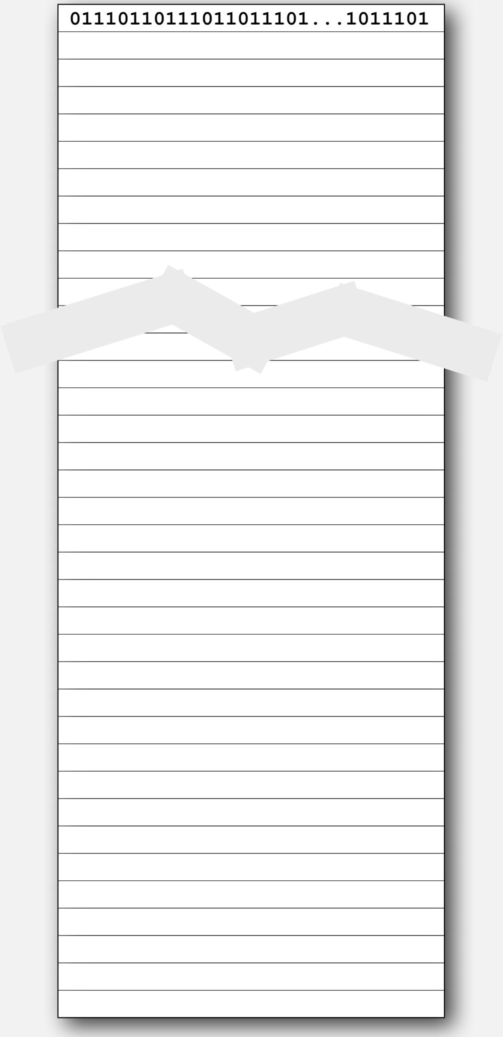
# String sorting challenge 2b

**Problem.** Sort huge array of random 128-bit numbers.

**Ex.** Supercomputer sort, internet router.

**Which sorting method to use?**

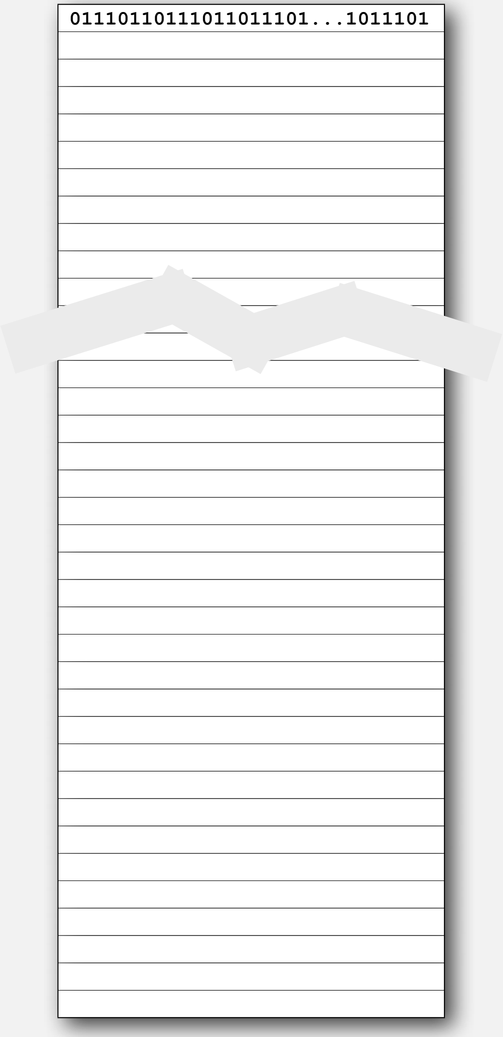
- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- LSD string sort.



# String sorting challenge 2b

**Problem.** Sort huge array of random 128-bit numbers.

**Ex.** Supercomputer sort, internet router.



**Which sorting method to use?**

- Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD string sort.

Divide each word into eight 16-bit “chars”
$2^{16}$ = 65,536 counters.
Sort in 8 passes.

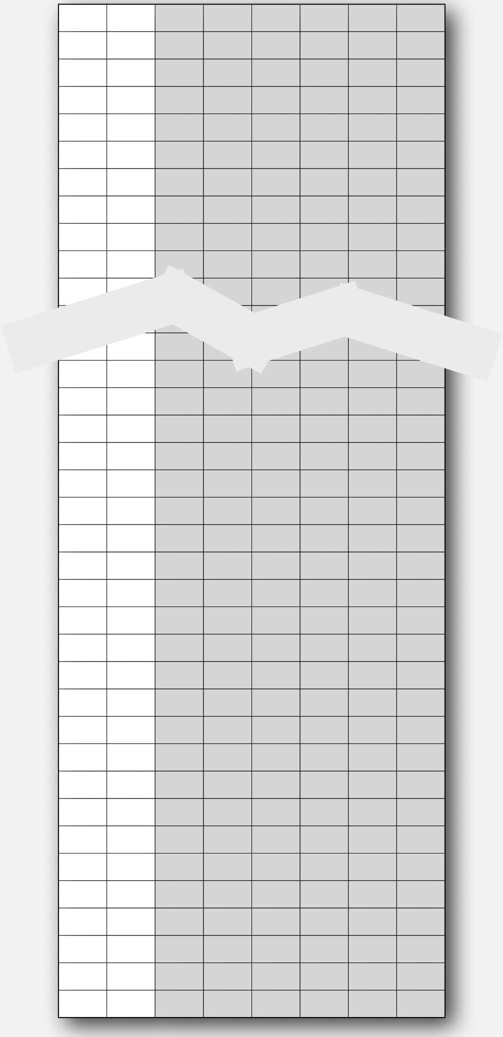
# String sorting challenge 2b

**Problem.** Sort huge array of random 128-bit numbers.

**Ex.** Supercomputer sort, internet router.

**Which sorting method to use?**

- ✓ Insertion sort.
- Mergesort.
- Quicksort.
- Heapsort.
- ✓ LSD string sort.



Divide each word into eight 16-bit "chars"
$2^{16}$ = 65,536 counters
LSD sort on leading 32 bits in 2 passes
Finish with insertion sort
Examines only ~25% of the data

# How to take a census in 1900s?

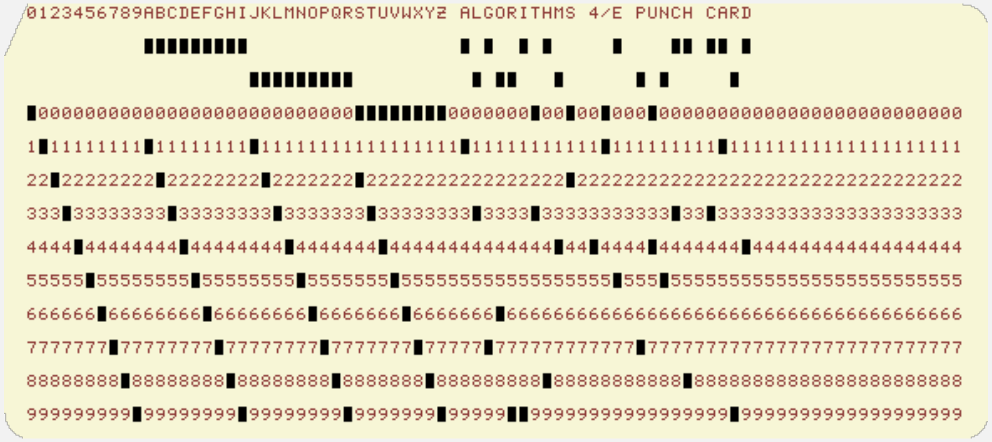
**1880 Census.** Took 1,500 people 7 years to manually process data.

U.S. DEPARTMENT OF COMMERCE BUREAU OF THE CENSUS

**Herman Hollerith.** Developed counting and sorting machine to automate.

- Use punch cards to record data (e.g., gender, age).
- Machine sorts one column at a time (into one of 12 bins).
- Typical question: how many women of age 20 to 30?

Hollerith tabulating machine and sorter



punch card (12 holes per column)

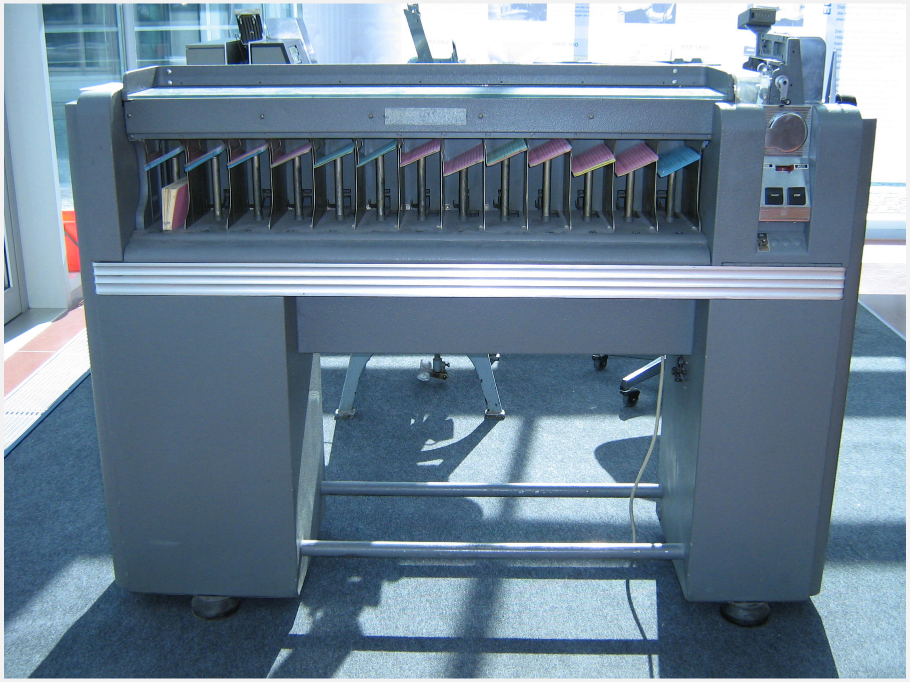
**1890 Census.** Finished months early and under budget!

# How to get rich sorting in 1900s?

**Punch cards.** [1900s to 1950s]

- Also useful for accounting, inventory, and business processes.
- Primary medium for data entry, storage, and processing.

Hollerith's company later merged with 3 others to form Computing Tabulating Recording Corporation (CTRC); the company was renamed in 1924.



IBM 80 Series Card Sorter (650 cards per minute)

IBM®

# LSD string sort: a moment in history (1960s)

**card punch**

**punched cards**

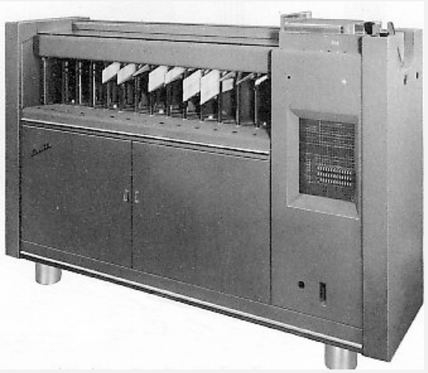
**card reader**

**mainframe**

**line printer**

To sort a card deck

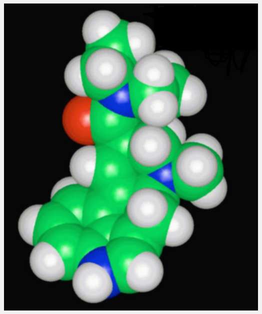
- start on right column
- put cards into hopper
- machine distributes into bins
- pick up cards (stable)
- move left one column
- continue until sorted



**card sorter**

not related to sorting

↓



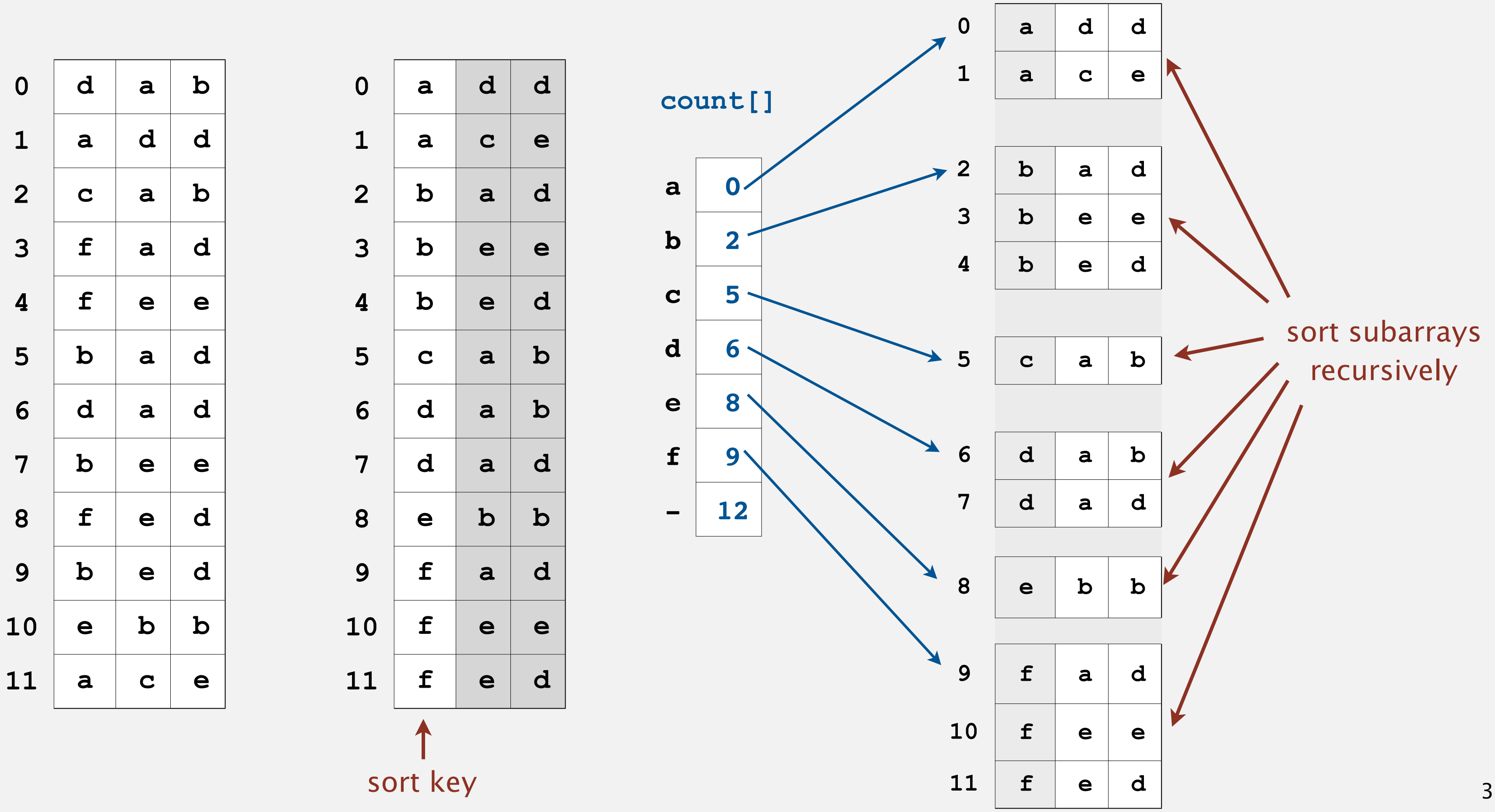
**Lysergic Acid Diethylamide**
**(Lucy in the Sky with Diamonds)**

- key-indexed counting
- LSD radix sort
- **MSD radix sort**
- 3-way radix quicksort
- suffix arrays

# Most-significant-digit-first string sort

## MSD string (radix) sort.

- Partition array into $R$ pieces according to first character (use key-indexed counting).
- Recursively sort all strings that start with each character (key-indexed counts delineate subarrays to sort).



| | | | |
|---|---|---|---|
| 0 | d | a | b |
| 1 | a | d | d |
| 2 | c | a | b |
| 3 | f | a | d |
| 4 | f | e | e |
| 5 | b | a | d |
| 6 | d | a | d |
| 7 | b | e | e |
| 8 | f | e | d |
| 9 | b | e | d |
| 10 | e | b | b |
| 11 | a | c | e |

| | | | |
|---|---|---|---|
| 0 | a | d | d |
| 1 | a | c | e |
| 2 | b | a | d |
| 3 | b | e | e |
| 4 | b | e | d |
| 5 | c | a | b |
| 6 | d | a | b |
| 7 | d | a | d |
| 8 | e | b | b |
| 9 | f | a | d |
| 10 | f | e | e |
| 11 | f | e | d |

sort key

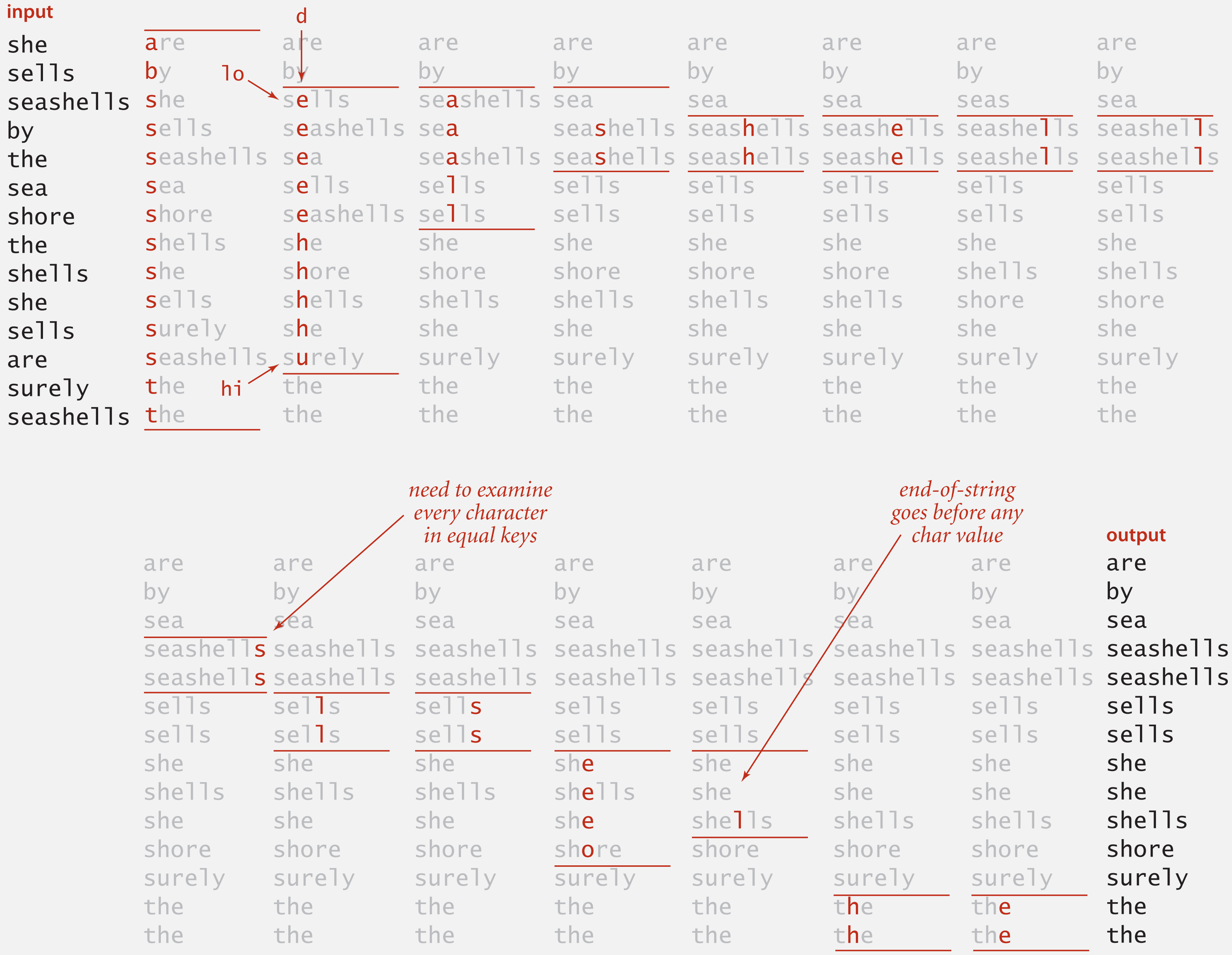
`count[]`

| | |
|---|---|
| a | 0 |
| b | 2 |
| c | 5 |
| d | 6 |
| e | 8 |
| f | 9 |
| - | 12 |

| | | | |
|---|---|---|---|
| 0 | a | d | d |
| 1 | a | c | e |
| 2 | b | a | d |
| 3 | b | e | e |
| 4 | b | e | d |
| 5 | c | a | b |
| 6 | d | a | b |
| 7 | d | a | d |
| 8 | e | b | b |
| 9 | f | a | d |
| 10 | f | e | e |
| 11 | f | e | d |

sort subarrays recursively

# MSD string sort: example



| input | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| she | are | are | are | are | are | are | are | are |
| sells | by | by | by | by | by | by | by | by |
| seashells | she | sells | seashells | sea | sea | sea | seas | sea |
| by | sells | seashells | sea | seashells | seashells | seashells | seashells | seashells |
| the | seashells | sea | seashells | seashells | seashells | seashells | seashells | seashells |
| sea | sea | sells | sells | sells | sells | sells | sells | sells |
| shore | shore | seashells | sells | sells | sells | sells | sells | sells |
| the | shells | she | she | she | she | she | she | she |
| shells | she | shore | shore | shore | shore | shore | shells | shells |
| she | sells | shells | shells | shells | shells | shells | shore | shore |
| sells | surely | she | she | she | she | she | she | she |
| are | seashells | surely | surely | surely | surely | surely | surely | surely |
| surely | the | the | the | the | the | the | the | the |
| seashells | the | the | the | the | the | the | the | the |

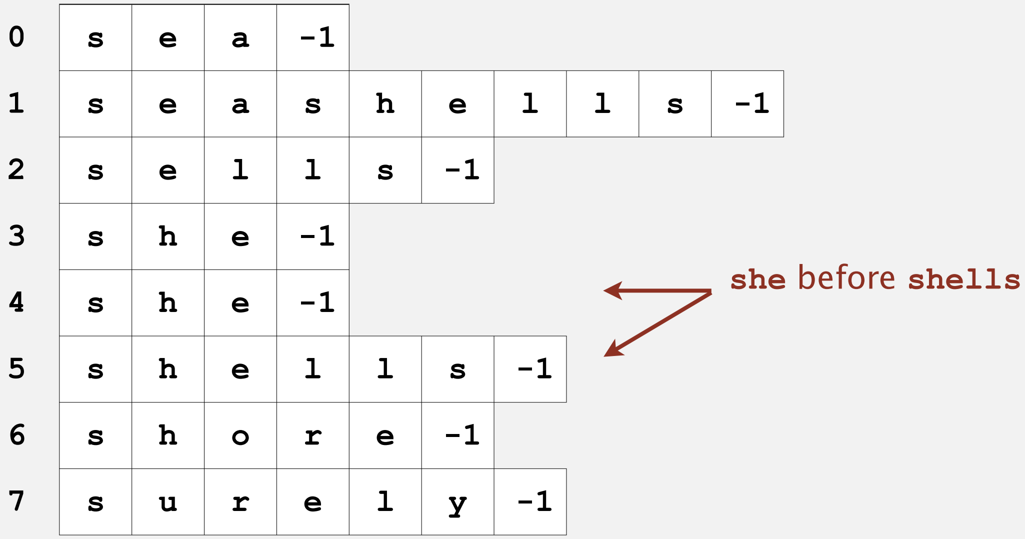
| | | | | | | | output |
|---|---|---|---|---|---|---|---|
| are | are | are | are | are | are | are | are |
| by | by | by | by | by | by | by | by |
| sea | sea | sea | sea | sea | sea | sea | sea |
| seashells | seashells | seashells | seashells | seashells | seashells | seashells | seashells |
| seashells | seashells | seashells | seashells | seashells | seashells | seashells | seashells |
| sells | sells | sells | sells | sells | sells | sells | sells |
| sells | sells | sells | sells | sells | sells | sells | sells |
| she | she | she | she | she | she | she | she |
| shells | shells | shells | shells | she | she | she | she |
| she | she | she | she | shells | shells | shells | shells |
| shore | shore | shore | shore | shore | shore | shore | shore |
| surely | surely | surely | surely | surely | surely | surely | surely |
| the | the | the | the | the | the | the | the |
| the | the | the | the | the | the | the | the |

Trace of recursive calls for MSD string sort (no cutoff for small subarrays, subarrays of size 0 and 1 omitted)

# Variable-length strings

Treat strings as if they had an extra char at end (smaller than any char).

why smaller?



| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | s | e | a | -1 | | | | | | |
| 1 | s | e | a | s | h | e | l | l | s | -1 |
| 2 | s | e | l | l | s | -1 | | | | |
| 3 | s | h | e | -1 | | | | | | |
| 4 | s | h | e | -1 | | | | | | |
| 5 | s | h | e | l | l | s | -1 | | | |
| 6 | s | h | o | r | e | -1 | | | | |
| 7 | s | u | r | e | l | y | -1 | | | |

`she` before `shells`

```
private static int charAt(String s, int d)
{
   if (d < s.length()) return s.charAt(d);
   else return -1;
}
```

C strings. Have extra char `'\0'` at end ⇒ no extra work needed.

# MSD string sort: Java implementation

```
public static void sort(String[] a)
{
   aux = new String[a.length];
   sort(a, aux, 0, a.length, 0);
}

private static void sort(String[] a, String[] aux, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int[] count = new int[R+2];
   for (int i = lo; i <= hi; i++)
      count[charAt(a[i], d) + 2]++;
   for (int r = 0; r < R+1; r++)
      count[r+1] += count[r];
   for (int i = lo; i <= hi; i++)
      aux[count[charAt(a[i], d) + 1]++] = a[i];
   for (int i = lo; i <= hi; i++)
      a[i] = aux[i - lo];

   for (int r = 0; r < R; r++)
      sort(a, aux, lo + count[r], lo + count[r+1] - 1, d+1);
}
```

can recycle `aux[]` array
but not `count[]` array

key-indexed counting

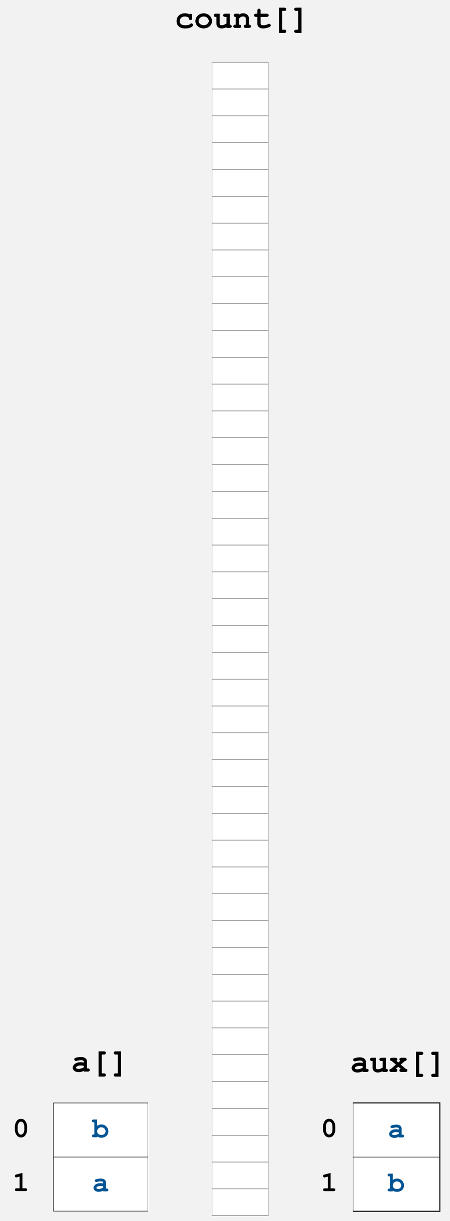
sort R subarrays recursively

# MSD string sort: potential for disastrous performance

Observation 1. Much too slow for small subarrays.

- Each function call needs its own `count[]` array.
- ASCII (256 counts): 100x slower than copy pass for $N = 2$.
- Unicode (65,536 counts): 32,000x slower for $N = 2$.

Observation 2. Huge number of small subarrays because of recursion.



# Cutoff to insertion sort

Solution. Cutoff to insertion sort for small subarrays.

- Insertion sort, but start at $d^{th}$ character.
- Implement `less()` so that it compares starting at $d^{th}$ character.

```
public static void sort(String[] a, int lo, int hi, int d)
{
   for (int i = lo; i <= hi; i++)
      for (int j = i; j > lo && less(a[j], a[j-1], d); j--)
         exch(a, j, j-1);
}
```

```
private static boolean less(String v, String w, int d)
{  return v.substring(d).compareTo(w.substring(d)) < 0;  }
```

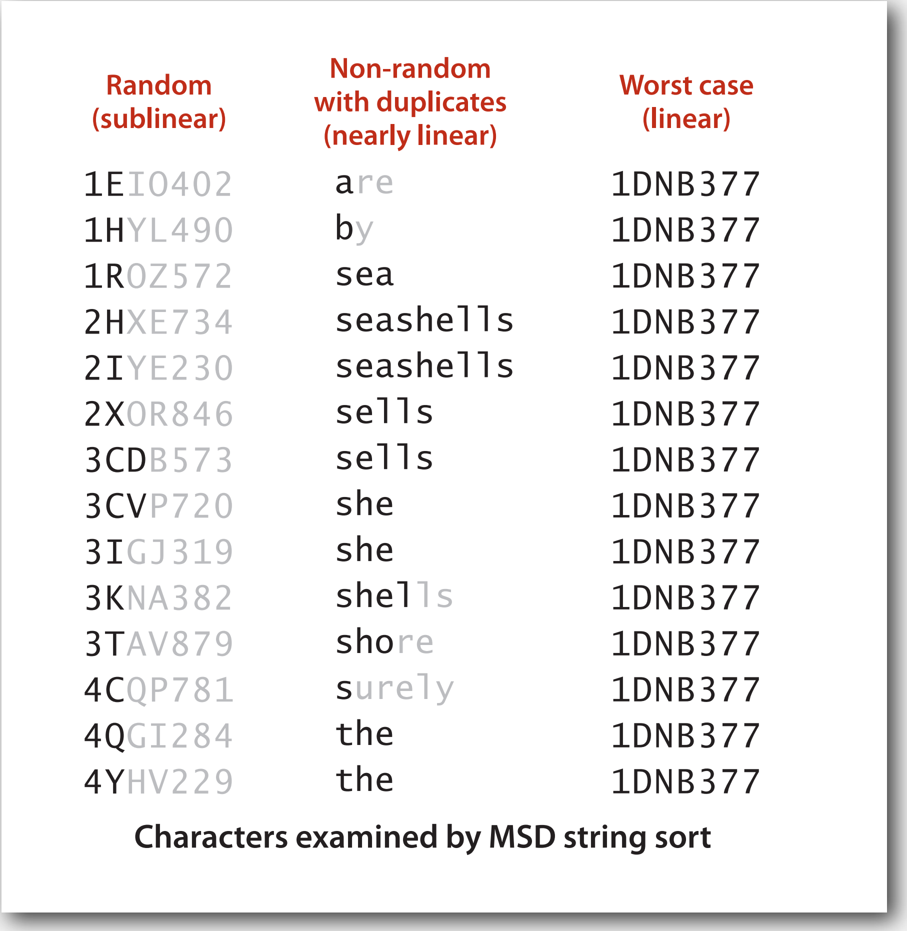
in Java, forming and comparing substrings is faster than directly comparing chars with charAt()

# MSD string sort: performance

## Number of characters examined.

- MSD examines just enough characters to sort the keys.
- Number of characters examined depends on keys.
- Can be sublinear in input size!

compareTo() based sorts
can also be sublinear!



| Random (sublinear) | Non-random with duplicates (nearly linear) | Worst case (linear) |
|---|---|---|
| 1EIO402 | are | 1DNB377 |
| 1HYL490 | by | 1DNB377 |
| 1ROZ572 | sea | 1DNB377 |
| 2HXE734 | seashells | 1DNB377 |
| 2IYE230 | seashells | 1DNB377 |
| 2XOR846 | sells | 1DNB377 |
| 3CDB573 | sells | 1DNB377 |
| 3CVP720 | she | 1DNB377 |
| 3IGJ319 | she | 1DNB377 |
| 3KNA382 | shells | 1DNB377 |
| 3TAV879 | shore | 1DNB377 |
| 4CQP781 | surely | 1DNB377 |
| 4QGI284 | the | 1DNB377 |
| 4YHV229 | the | 1DNB377 |

Characters examined by MSD string sort

# Summary of the performance of sorting algorithms

Frequency of operations.

| algorithm | guarantee | random | extra space | stable? | operations on keys |
|---|---|---|---|---|---|
| insertion sort | $N^2 / 2$ | $N^2 / 4$ | 1 | yes | `compareTo()` |
| mergesort | N lg N | N lg N | N | yes | `compareTo()` |
| quicksort | 1.39 N lg N * | 1.39 N lg N | c lg N | no | `compareTo()` |
| heapsort | 2 N lg N | 2 N lg N | 1 | no | `compareTo()` |
| LSD † | 2 N W | 2 N W | N + R | yes | `charAt()` |
| MSD ‡ | 2 N W | $N \log_R N$ | N + D R | yes | `charAt()` |

D = function-call stack depth
(length of longest prefix match)

* probabilistic
† fixed-length W keys
‡ average-length W keys

# MSD string sort vs. quicksort for strings

## Disadvantages of MSD string sort.

- Accesses memory "randomly" (cache inefficient).
- Inner loop has a lot of instructions.
- Extra space for `count[]`.
- Extra space for `aux[]`.

## Disadvantage of quicksort.

- Linearithmic number of string compares (not linear).
- Has to rescan many characters in keys with long prefix matches.

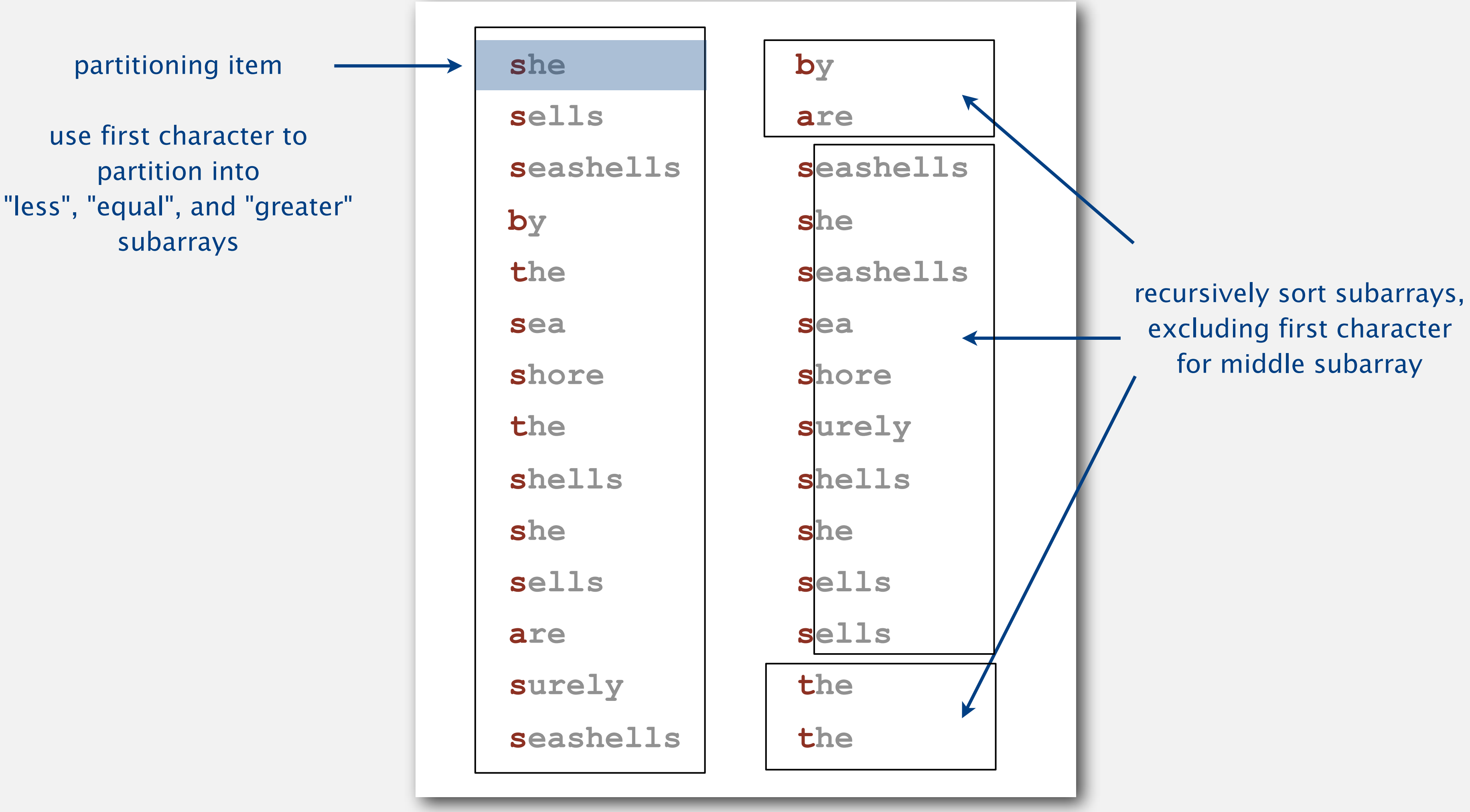
Goal. Combine advantages of MSD and quicksort.

- key-indexed counting
- LSD radix sort
- MSD radix sort
- **3-way radix quicksort**
- suffix arrays

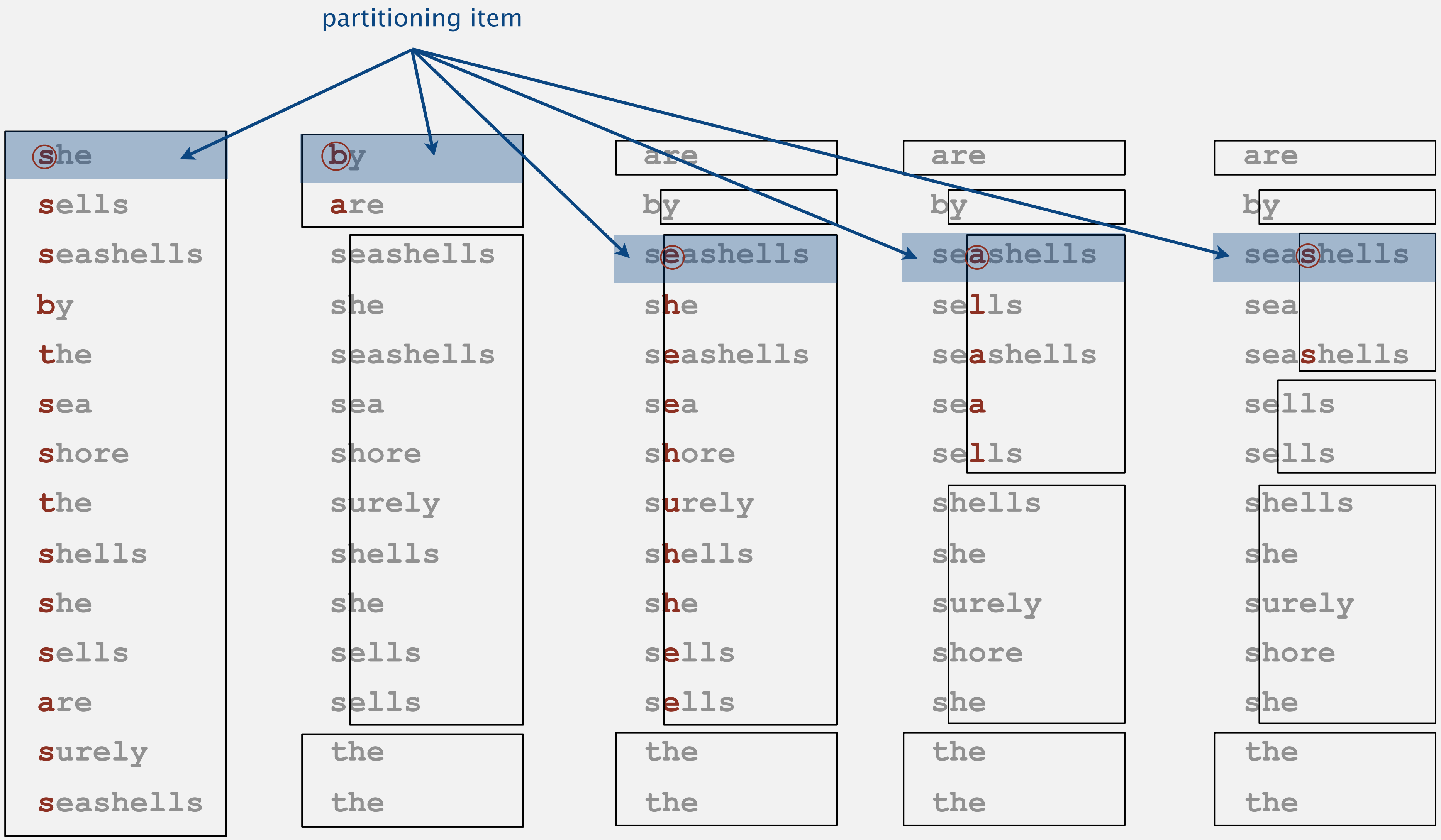
# 3-way string quicksort (Bentley and Sedgewick, 1997)

**Overview.** Do 3-way partitioning on the $d^{th}$ character.

- Less overhead than $R$-way partitioning in MSD string sort.
- Does not re-examine characters equal to the partitioning char (but does re-examine characters not equal to the partitioning char).



# 3-way string quicksort: trace of recursive calls



| | | | | |
|---|---|---|---|---|
| she | by | are | are | are |
| sells | are | by | by | by |
| seashells | seashells | seashells | seashells | seashells |
| by | she | she | sells | sea |
| the | seashells | seashells | seashells | seashells |
| sea | sea | sea | sea | sells |
| shore | shore | shore | sells | sells |
| the | surely | surely | shells | shells |
| shells | shells | shells | she | she |
| she | she | she | surely | surely |
| sells | sells | sells | shore | shore |
| are | sells | sells | she | she |
| surely | the | the | the | the |
| seashells | the | the | the | the |

Trace of first few recursive calls for 3-way string quicksort (subarrays of size 1 not shown)

# 3-way string quicksort: Java implementation

```
private static void sort(String[] a)
{  sort(a, 0, a.length - 1, 0);  }

private static void sort(String[] a, int lo, int hi, int d)
{
   if (hi <= lo) return;
   int lt = lo, gt = hi;
   int v = charAt(a[lo], d);
   int i = lo + 1;
   while (i <= gt)
   {
      int t = charAt(a[i], d);
      if      (t < v) exch(a, lt++, i++);
      else if (t > v) exch(a, i, gt--);
      else            i++;
   }

   sort(a, lo, lt-1, d);
   if (v >= 0) sort(a, lt, gt, d+1);
   sort(a, gt+1, hi, d);
}
```

3-way partitioning
(using d<sup>th</sup> character)

to handle variable-length strings

sort 3 subarrays recursively

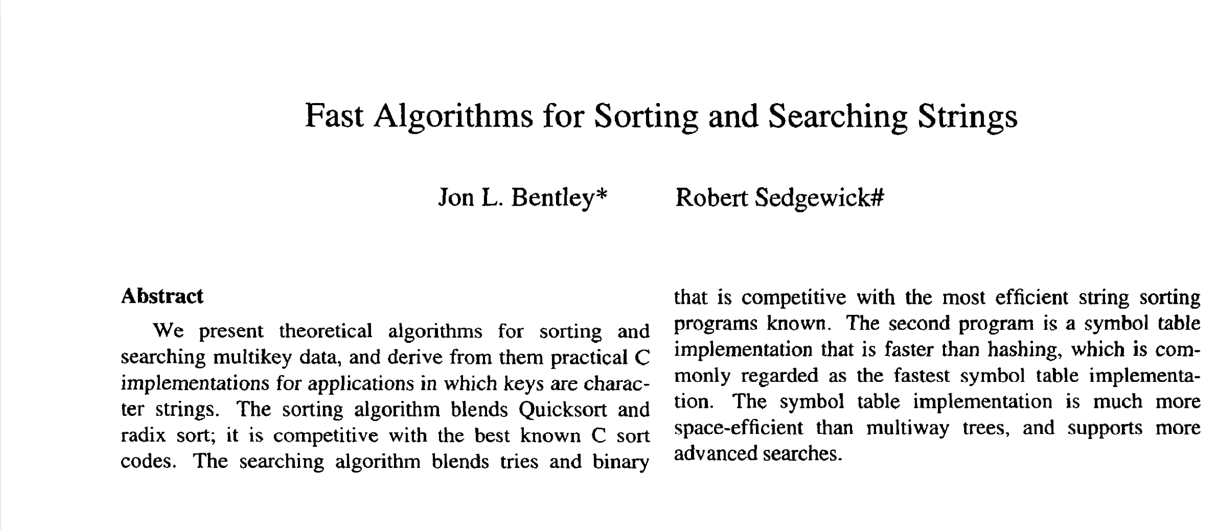
# 3-way string quicksort vs. standard quicksort

## Standard quicksort.

- Uses $\sim 2 N \ln N$ string compares on average.
- Costly for keys with long common prefixes (and this is a common case!)

## 3-way string (radix) quicksort.

- Uses $\sim 2 N \ln N$ character compares on average for random strings.
- Avoids re-comparing long common prefixes.



Fast Algorithms for Sorting and Searching Strings

Jon L. Bentley* Robert Sedgewick#

**Abstract**

We present theoretical algorithms for sorting and searching multikey data, and derive from them practical C implementations for applications in which keys are character strings. The sorting algorithm blends Quicksort and radix sort; it is competitive with the best known C sort codes. The searching algorithm blends tries and binary that is competitive with the most efficient string sorting programs known. The second program is a symbol table implementation that is faster than hashing, which is commonly regarded as the fastest symbol table implementation. The symbol table implementation is much more space-efficient than multiway trees, and supports more advanced searches.

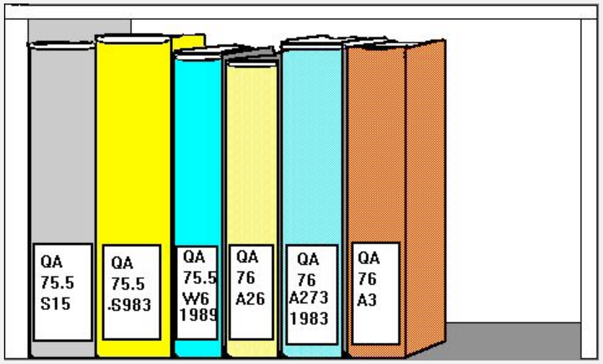
# 3-way string quicksort vs. MSD string sort

MSD string sort.

- Is cache-inefficient.
- Too much memory storing `count[]`.
- Too much overhead reinitializing `count[]` and `aux[]`.

3-way string quicksort.

- Has a short inner loop.
- Is cache-friendly.
- Is in-place.



library of Congress call numbers

Bottom line. 3-way string quicksort is the method of choice for sorting strings.

# Summary of the performance of sorting algorithms

Frequency of operations.

| algorithm | guarantee | random | extra space | stable? | operations on keys |
|---|---|---|---|---|---|
| insertion sort | $N^2 / 2$ | $N^2 / 4$ | 1 | yes | `compareTo()` |
| mergesort | N lg N | N lg N | N | yes | `compareTo()` |
| quicksort | 1.39 N lg N * | 1.39 N lg N | c lg N | no | `compareTo()` |
| heapsort | 2 N lg N | 2 N lg N | 1 | no | `compareTo()` |
| LSD † | 2 N W | 2 N W | N + R | yes | `charAt()` |
| MSD ‡ | 2 N W | $N \log_R N$ | N + D R | yes | `charAt()` |
| 3-way string quicksort | 1.39 W N lg N * | 1.39 N lg N | log N + W | no | `charAt()` |

* probabilistic
† fixed-length W keys
‡ average-length W keys

- key-indexed counting
- LSD radix sort
- MSD radix sort
- 3-way radix quicksort
- **suffix arrays**

# Longest repeated substring

Given a string of $N$ characters, find the longest repeated substring.

```
a a c a a g t t t a c a a g c a t g a t g c t g t a c t a
g g a g a g t t a t a c t g g t c g t c a a a c c t g a a
c c t a a t c c t t g t g t g t a c a c a c a c t a c t a
c t g t c g t c g t c a t a t a t c g a g a t c a t c g a
a c c g g a a g g c c g g a c a a g g c g g g g g g t a t
a g a t a g a t a g a c c c c t a g a t a c a c a t a c a
t a g a t c t a g c t a g c t a g c t c a t c g a t a c a
c a c t c t c a c a c t c a a g a g t t a t a c t g g t c
a a c a c a c t a c t a c g a c a g a c g a c c a a c c a
g a c a g a a a a a a a a c t c t a t a t c t a t a a a a
```

**Applications.** Bioinformatics, cryptanalysis, data compression, ...

# Longest repeated substring: a musical application

Visualize repetitions in music. `http://www.bewitched.com`

**Mary Had a Little Lamb**

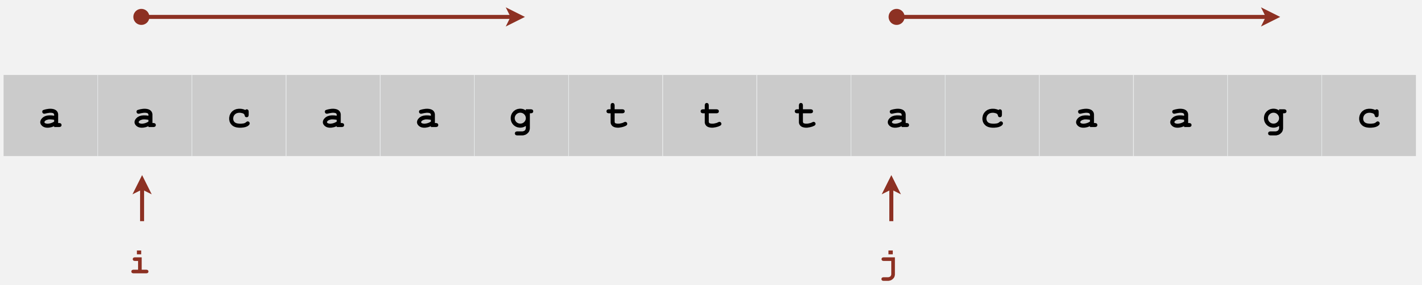
**Bach's Goldberg Variations**

# Longest repeated substring

Given a string of $N$ characters, find the longest repeated substring.

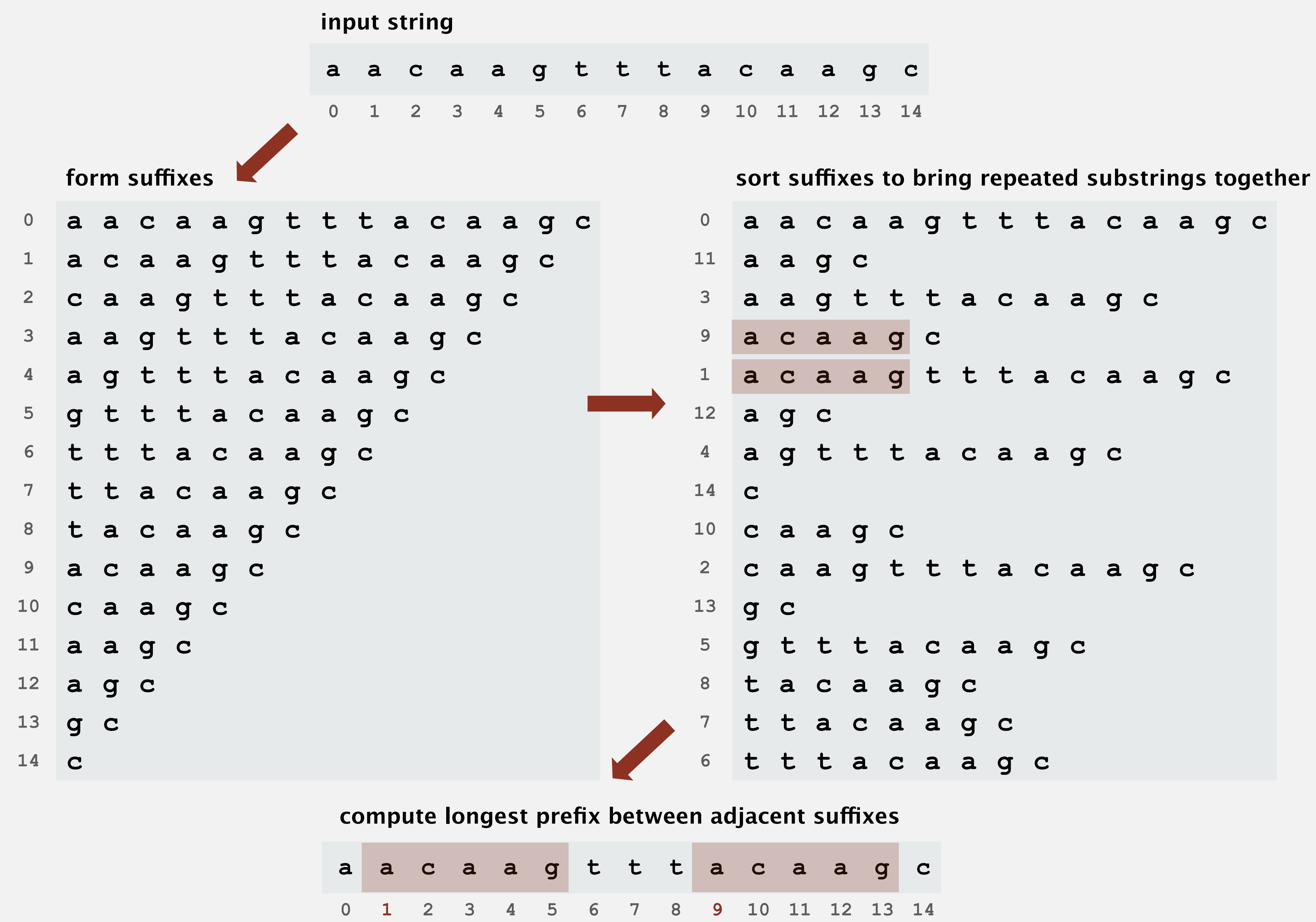
Brute-force algorithm.

- Try all indices $i$ and $j$ for start of possible match.
- Compute longest common prefix (LCP) for each pair.



Analysis. Running time ≤ $D N^2$, where $D$ is length of longest match.

# Longest repeated substring: a sorting solution



**input string**

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | a | c | a | a | g | t | t | t | a | c | a | a | g | c |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

**form suffixes**

```
 0  a a c a a g t t t a c a a g c
 1  a c a a g t t t a c a a g c
 2  c a a g t t t a c a a g c
 3  a a g t t t a c a a g c
 4  a g t t t a c a a g c
 5  g t t t a c a a g c
 6  t t t a c a a g c
 7  t t a c a a g c
 8  t a c a a g c
 9  a c a a g c
10  c a a g c
11  a a g c
12  a g c
13  g c
14  c
```

**sort suffixes to bring repeated substrings together**

```
 0  a a c a a g t t t a c a a g c
11  a a g c
 3  a a g t t t a c a a g c
 9  a c a a g c
 1  a c a a g t t t a c a a g c
12  a g c
 4  a g t t t a c a a g c
14  c
10  c a a g c
 2  c a a g t t t a c a a g c
13  g c
 5  g t t t a c a a g c
 8  t a c a a g c
 7  t t a c a a g c
 6  t t t a c a a g c
```

**compute longest prefix between adjacent suffixes**

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | a | c | a | a | g | t | t | t | a | c | a | a | g | c |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |

# Longest repeated substring: Java implementation

```
public String lrs(String s)
{
  int N = s.length();

  String[] suffixes = new String[N];
  for (int i = 0; i < N; i++)
     suffixes[i] = s.substring(i, N);

  Arrays.sort(suffixes);

  String lrs = "";
  for (int i = 0; i < N-1; i++)
  {
     int len = lcp(suffixes[i], suffixes[i+1]);
     if (len > lrs.length())
        lrs = suffixes[i].substring(0, len);
  }
  return lrs;
}
```

create suffixes
(linear time and space)

sort suffixes

find LCP between
adjacent suffixes in
sorted order

```
% java LRS < mobydick.txt
,- Such a funny, sporty, gamy, jesty, joky, hoky-poky lad, is the Ocean, oh! Th
```

# Longest repeated substring: empirical analysis

| input file | characters | brute | suffix sort | length of LRS |
|---|---|---|---|---|
| `LRS.java` | 2,162 | 0.6 sec | 0.14 sec | 73 |
| `amendments.txt` | 18,369 | 37 sec | 0.25 sec | 216 |
| `aesop.txt` | 191,945 | 1.2 hours | 1.0 sec | 58 |
| `mobydick.txt` | 1.2 million | 43 hours † | 7.6 sec | 79 |
| `chromosome11.txt` | 7.1 million | 2 months † | 61 sec | 12,567 |
| `pi.txt` | 10 million | 4 months † | 84 sec | 14 |
| `pipi.txt` | 20 million | forever † | ??? | 10 million |

† estimated

# Suffix sorting: worst-case input

Bad input: longest repeated substring very long.

- Ex: same letter repeated $N$ times.
- Ex: two copies of the same Java codebase.

**form suffixes**

| | |
|---|---|
| 0 | a a a a a a a a a a |
| 1 | a a a a a a a a a |
| 2 | a a a a a a a a |
| 3 | a a a a a a a |
| 4 | a a a a a a |
| 5 | a a a a a |
| 6 | a a a a |
| 7 | a a a |
| 8 | a a |
| 9 | a |

**sorted suffixes**

| | |
|---|---|
| 9 | a |
| 8 | a a |
| 7 | a a a |
| 6 | a a a a |
| 5 | a a a a a |
| 4 | a a a a a a |
| 3 | a a a a a a a |
| 2 | a a a a a a a a |
| 1 | a a a a a a a a a |
| 0 | a a a a a a a a a a |

Running time. Quadratic (or worse) in the length of the longest match.

# Suffix sorting challenge

**Problem.** Suffix sort an arbitrary string of length $N$.

**Q.** What is worst-case running time of best algorithm for problem?

- Quadratic.
- ✓ Linearithmic. ← Manber's algorithm
- ✓ Linear. ← suffix trees (see COS 423)
- Nobody knows.

# Suffix sorting in linearithmic time

**Manber's MSD algorithm overview.**

- Phase $0$: sort on first character using key-indexed counting sort.
- Phase $i$: given array of suffixes sorted on first $2^{i-1}$ characters, create array of suffixes sorted on first $2^{i}$ characters.

**Worst-case running time.** $N \lg N$.

- Finishes after $\lg N$ phases.
- Can perform a phase in linear time. (!) [ahead]

# Linearithmic suffix sort example:  phase 0

**original suffixes**

| | |
|---|---|
| 0 | b a b a a a a b c b a b a a a a a 0 |
| 1 | a b a a a a b c b a b a a a a a 0 |
| 2 | b a a a a b c b a b a a a a a 0 |
| 3 | a a a a b c b a b a a a a a 0 |
| 4 | a a a b c b a b a a a a a 0 |
| 5 | a a b c b a b a a a a a 0 |
| 6 | a b c b a b a a a a a 0 |
| 7 | b c b a b a a a a a 0 |
| 8 | c b a b a a a a a 0 |
| 9 | b a b a a a a a 0 |
| 10 | a b a a a a a 0 |
| 11 | b a a a a a 0 |
| 12 | a a a a a 0 |
| 13 | a a a a 0 |
| 14 | a a a 0 |
| 15 | a a 0 |
| 16 | a 0 |
| 17 | 0 |

**key-indexed counting sort (first character)**

| | | |
|---|---|---|
| 17 | 0 | |
| 1 | a | b a a a a b c b a b a a a a a 0 |
| 16 | a | 0 |
| 3 | a | a a a b c b a b a a a a a 0 |
| 4 | a | a a b c b a b a a a a a 0 |
| 5 | a | a b c b a b a a a a a 0 |
| 6 | a | b c b a b a a a a a 0 |
| 15 | a | a 0 |
| 14 | a | a a 0 |
| 13 | a | a a a 0 |
| 12 | a | a a a a 0 |
| 10 | a | b a a a a a 0 |
| 0 | b | a b a a a a b c b a b a a a a a 0 |
| 9 | b | a b a a a a a 0 |
| 11 | b | a a a a a 0 |
| 7 | b | c b a b a a a a a 0 |
| 2 | b | a a a a b c b a b a a a a a 0 |
| 8 | c | b a b a a a a a 0 |

↑

*sorted*

# Linearithmic suffix sort example: phase 1

**original suffixes**

| | |
|---|---|
| 0 | b a b a a a a b c b a b a a a a a 0 |
| 1 | a b a a a a b c b a b a a a a a 0 |
| 2 | b a a a a b c b a b a a a a a 0 |
| 3 | a a a a b c b a b a a a a a 0 |
| 4 | a a a b c b a b a a a a a 0 |
| 5 | a a b c b a b a a a a a 0 |
| 6 | a b c b a b a a a a a 0 |
| 7 | b c b a b a a a a a 0 |
| 8 | c b a b a a a a a 0 |
| 9 | b a b a a a a a 0 |
| 10 | a b a a a a a 0 |
| 11 | b a a a a a 0 |
| 12 | a a a a a 0 |
| 13 | a a a a 0 |
| 14 | a a a 0 |
| 15 | a a 0 |
| 16 | a 0 |
| 17 | 0 |

**index sort (first two characters)**

| | | |
|---|---|---|
| 17 | 0 | |
| 16 | a 0 | |
| 12 | a a | a a a 0 |
| 3 | a a | a a b c b a b a a a a a 0 |
| 4 | a a | a b c b a b a a a a a 0 |
| 5 | a a | b c b a b a a a a a 0 |
| 13 | a a | a a 0 |
| 15 | a a | 0 |
| 14 | a a | a 0 |
| 6 | a b | c b a b a a a a a 0 |
| 1 | a b | a a a a b c b a b a a a a a 0 |
| 10 | a b | a a a a a 0 |
| 0 | b a | b a a a a b c b a b a a a a a 0 |
| 9 | b a | b a a a a a 0 |
| 11 | b a | a a a a 0 |
| 2 | b a | a a a b c b a b a a a a a 0 |
| 7 | b c | b a b a a a a a 0 |
| 8 | c b | a b a a a a a 0 |

↑ *sorted*

# Linearithmic suffix sort example:  phase 2

**original suffixes**

| | |
|---|---|
| 0 | b a b a a a a b c b a b a a a a a 0 |
| 1 | a b a a a a b c b a b a a a a a 0 |
| 2 | b a a a a b c b a b a a a a a 0 |
| 3 | a a a a b c b a b a a a a a 0 |
| 4 | a a a b c b a b a a a a a 0 |
| 5 | a a b c b a b a a a a a 0 |
| 6 | a b c b a b a a a a a 0 |
| 7 | b c b a b a a a a a 0 |
| 8 | c b a b a a a a a 0 |
| 9 | b a b a a a a a 0 |
| 10 | a b a a a a a 0 |
| 11 | b a a a a a 0 |
| 12 | a a a a a 0 |
| 13 | a a a a 0 |
| 14 | a a a 0 |
| 15 | a a 0 |
| 16 | a 0 |
| 17 | 0 |

**index sort (first four characters)**

| | | |
|---|---|---|
| 17 | 0 | |
| 16 | a 0 | |
| 15 | a a 0 | |
| 14 | a a a 0 | |
| 3 | a a a a | b c b a b a a a a a 0 |
| 12 | a a a a | a 0 |
| 13 | a a a a | 0 |
| 4 | a a a b | c b a b a a a a a 0 |
| 5 | a a b c | b a b a a a a a 0 |
| 1 | a b a a | a a b c b a b a a a a a 0 |
| 10 | a b a a | a a a 0 |
| 6 | a b c b | a b a a a a a 0 |
| 2 | b a a a | a b c b a b a a a a a 0 a 0 |
| 11 | b a a a | a a 0 |
| 0 | b a b a | a a a b c b a b a a a a a 0 |
| 9 | b a b a | a a a a 0 |
| 7 | b c b a | b a a a a a 0 |
| 8 | c b a b | a a a a a 0 |

↑
*sorted*

# Linearithmic suffix sort example: phase 3

**original suffixes**

| | |
|---|---|
| 0 | b a b a a a a b c b a b a a a a a 0 |
| 1 | a b a a a a b c b a b a a a a a 0 |
| 2 | b a a a a b c b a b a a a a a 0 |
| 3 | a a a a b c b a b a a a a a 0 |
| 4 | a a a b c b a b a a a a a 0 |
| 5 | a a b c b a b a a a a a 0 |
| 6 | a b c b a b a a a a a 0 |
| 7 | b c b a b a a a a a 0 |
| 8 | c b a b a a a a a 0 |
| 9 | b a b a a a a a 0 |
| 10 | a b a a a a a 0 |
| 11 | b a a a a a 0 |
| 12 | a a a a a 0 |
| 13 | a a a a 0 |
| 14 | a a a 0 |
| 15 | a a 0 |
| 16 | a 0 |
| 17 | 0 |

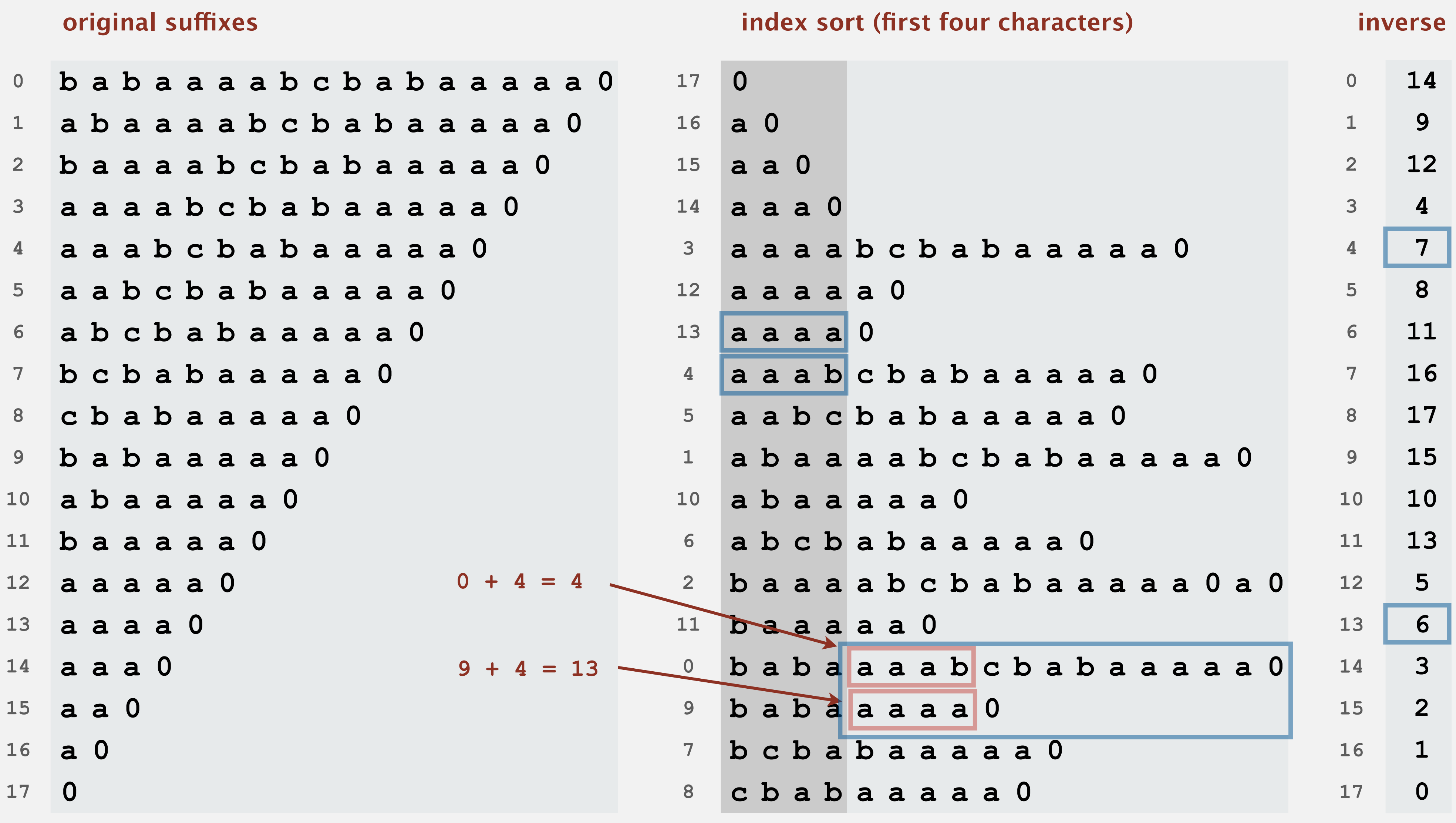
**index sort (first eight characters)**

| | |
|---|---|
| 17 | 0 |
| 16 | a 0 |
| 15 | a a 0 |
| 14 | a a a 0 |
| 13 | a a a a 0 |
| 12 | a a a a a 0 |
| 3 | a a a a b c b a b a a a a a 0 |
| 4 | a a a b c b a b a a a a a 0 |
| 5 | a a b c b a b a a a a a 0 |
| 10 | a b a a a a a 0 |
| 1 | a b a a a a b c b a b a a a a a 0 |
| 6 | a b c b a b a a a a a 0 |
| 11 | b a a a a a 0 |
| 2 | b a a a a b c b a b a a a a a 0 a 0 |
| 9 | b a b a a a a a 0 |
| 0 | b a b a a a a b c b a b a a a a a 0 |
| 7 | b c b a b a a a a a 0 |
| 8 | c b a b a a a a a 0 |

↑

finished (no equal keys)

# Constant-time string compare by indexing into inverse



original suffixes

| | |
|---|---|
| 0 | b a b a a a a b c b a b a a a a a 0 |
| 1 | a b a a a a b c b a b a a a a a 0 |
| 2 | b a a a a b c b a b a a a a a 0 |
| 3 | a a a a b c b a b a a a a a 0 |
| 4 | a a a b c b a b a a a a a 0 |
| 5 | a a b c b a b a a a a a 0 |
| 6 | a b c b a b a a a a a 0 |
| 7 | b c b a b a a a a a 0 |
| 8 | c b a b a a a a a 0 |
| 9 | b a b a a a a a 0 |
| 10 | a b a a a a a 0 |
| 11 | b a a a a a 0 |
| 12 | a a a a a 0 |
| 13 | a a a a 0 |
| 14 | a a a 0 |
| 15 | a a 0 |
| 16 | a 0 |
| 17 | 0 |

index sort (first four characters)

| | |
|---|---|
| 17 | 0 |
| 16 | a 0 |
| 15 | a a 0 |
| 14 | a a a 0 |
| 3 | a a a a b c b a b a a a a a 0 |
| 12 | a a a a a 0 |
| 13 | a a a a 0 |
| 4 | a a a b c b a b a a a a a 0 |
| 5 | a a b c b a b a a a a a 0 |
| 1 | a b a a a a b c b a b a a a a a 0 |
| 10 | a b a a a a a 0 |
| 6 | a b c b a b a a a a a 0 |
| 2 | b a a a a b c b a b a a a a a 0 a 0 |
| 11 | b a a a a a 0 |
| 0 | b a b a a a a b c b a b a a a a a 0 |
| 9 | b a b a a a a a 0 |
| 7 | b c b a b a a a a a 0 |
| 8 | c b a b a a a a a 0 |

inverse

| | |
|---|---|
| 0 | 14 |
| 1 | 9 |
| 2 | 12 |
| 3 | 4 |
| 4 | 7 |
| 5 | 8 |
| 6 | 11 |
| 7 | 16 |
| 8 | 17 |
| 9 | 15 |
| 10 | 10 |
| 11 | 13 |
| 12 | 5 |
| 13 | 6 |
| 14 | 3 |
| 15 | 2 |
| 16 | 1 |
| 17 | 0 |

$\texttt{suffixes}_4\texttt{[13]} \leq \texttt{suffixes}_4\texttt{[4]}$ (*because* `inverse[13]` < `inverse[4]`)

*so* $\texttt{suffixes}_8\texttt{[9]} \leq \texttt{suffixes}_8\texttt{[0]}$

# Keyword-in-context search

Given a text of $N$ characters, preprocess it to enable fast substring search (find all occurrences of query string context).

```
% java KWIC tale.txt 15
search
o st giless to search for contraband
her unavailing search for your fathe
le and gone in search of her husband
t provinces in search of impoverishe
 dispersing in search of other carri
n that bed and search the straw hold

better thing
t is a far far better thing that i do than
 some sense of better things else forgotte
was capable of better things mr carton ent
```

characters of surrounding context

**Applications.** Linguistics, databases, web search, word processing, ….

# Keyword-in-context search: suffix-sorting solution

- Preprocess: suffix sort the text.
- Query: binary search for query; scan until mismatch.

**KWIC search for "search" in Tale of Two Cities**

```
                     ⋮
632698   sealed_my_letter_and_...
713727   seamstress_is_lifted_...
660598   seamstress_of_twenty_...
67610    seamstress_who_was_wi...
4430     search_for_contraband...
42705    search_for_your_fathe...
499797   search_of_her_husband...
182045   search_of_impoverishe...
143399   search_of_other_carri...
411801   search_the_straw_hold...
158410   seared_marking_about_...
691536   seas_and_madame_defar...
536569   sease_a_terrible_pass...
484763   sease_that_had_brough...
                     ⋮
```

# String sorting summary

We can develop linear-time sorts.

- Key compares not necessary for string keys.
- Use characters as index in an array.

We can develop sublinear-time sorts.

- Should measure amount of data in keys, not number of keys.
- Not all of the data has to be examined.

3-way string quicksort is asymptotically optimal.

- $1.39\, N \lg N$ chars for random data.

Long strings are rarely random in practice.

- Goal is often to learn the structure!
- May need specialized algorithms.