

# 2.1 ELEMENTARY SORTS



- ▶ rules of the game
- ▶ selection sort
- ▶ insertion sort
- ▶ shellsort
- ▶ shuffling
- ▶ convex hull

- ▶ **rules of the game**

- ▶ selection sort

- ▶ insertion sort

- ▶ shellsort

- ▶ shuffling

- ▶ convex hull

## Sorting problem

Ex. Student records in a university.

	Chen	3	A	991-878-4944	308 Blair
	Rohde	2	A	232-343-5555	343 Forbes
	Gazsi	4	B	766-093-9873	101 Brown
item →	<b>Furia</b>	<b>1</b>	<b>A</b>	<b>766-093-9873</b>	<b>101 Brown</b>
	Kanaga	3	B	898-122-9643	22 Brown
	Andrews	3	A	664-480-0023	097 Little
key →	<b>Battle</b>	<b>4</b>	<b>C</b>	<b>874-088-1212</b>	<b>121 Whitman</b>

Sort. Rearrange array of  $N$  items into ascending order.

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

## Sample sort client

Goal. Sort **any** type of data.

Ex 1. Sort random real numbers in ascending order.

seems artificial, but stay tuned for an application

```
public class Experiment
{
    public static void main(String[] args)
    {
        int N = Integer.parseInt(args[0]);
        Double[] a = new Double[N];
        for (int i = 0; i < N; i++)
            a[i] = StdRandom.uniform();
        Insertion.sort(a);
        for (int i = 0; i < N; i++)
            StdOut.println(a[i]);
    }
}
```

```
% java Experiment 10
0.08614716385210452
0.09054270895414829
0.10708746304898642
0.21166190071646818
0.363292849257276
0.460954145685913
0.5340026311350087
0.7216129793703496
0.9003500354411443
0.9293994908845686
```

## Sample sort client

Goal. Sort **any** type of data.

Ex 2. Sort strings from file in alphabetical order.

```
public class StringSorter
{
    public static void main(String[] args)
    {
        String[] a = In.readStrings(args[0]);
        Insertion.sort(a);
        for (int i = 0; i < a.length; i++)
            StdOut.println(a[i]);
    }
}
```

```
% more words3.txt
bed bug dad yet zoo ... all bad yes

% java StringSorter words3.txt
all bad bed bug dad ... yes yet zoo
```

## Sample sort client

Goal. Sort **any** type of data.

Ex 3. Sort the files in a given directory by filename.

```
import java.io.File;
public class FileSorter
{
    public static void main(String[] args)
    {
        File directory = new File(args[0]);
        File[] files = directory.listFiles();
        Insertion.sort(files);
        for (int i = 0; i < files.length; i++)
            StdOut.println(files[i].getName());
    }
}
```

```
% java FileSorter .
Insertion.class
Insertion.java
InsertionX.class
InsertionX.java
Selection.class
Selection.java
Shell.class
Shell.java
ShellX.class
ShellX.java
```

## Callbacks

Goal. Sort **any** type of data.

Q. How can `sort()` know how to compare data of type `Double`, `String`, and `java.io.File` without any information about the type of an item's key?

Callback = reference to executable code.

- Client passes array of objects to `sort()` function.
- The `sort()` function calls back object's `compareTo()` method as needed.

Implementing callbacks.

- Java: **interfaces**.
- C: function pointers.
- C++: class-type functors.
- C#: delegates.
- Python, Perl, ML, Javascript: first-class functions.

# Callbacks: roadmap

## client

```
import java.io.File;
public class FileSorter
{
    public static void main(String[] args)
    {
        File directory = new File(args[0]);
        File[] files = directory.listFiles();
        Insertion.sort(files);
        for (int i = 0; i < files.length; i++)
            StdOut.println(files[i].getName());
    }
}
```

## object implementation

```
public class File
implements Comparable<File>
{
    ...
    public int compareTo(File b)
    {
        ...
        return -1;
        ...
        return +1;
        ...
        return 0;
    }
}
```

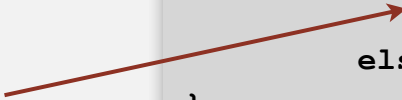
## Comparable interface (built in to Java)

```
public interface Comparable<Item>
{
    public int compareTo(Item that);
}
```

## sort implementation

```
public static void sort(Comparable[] a)
{
    int N = a.length;
    for (int i = 0; i < N; i++)
        for (int j = i; j > 0; j--)
            if (a[j].compareTo(a[j-1]) < 0)
                exch(a, j, j-1);
            else break;
}
```

key point: no dependence  
on `File` data type





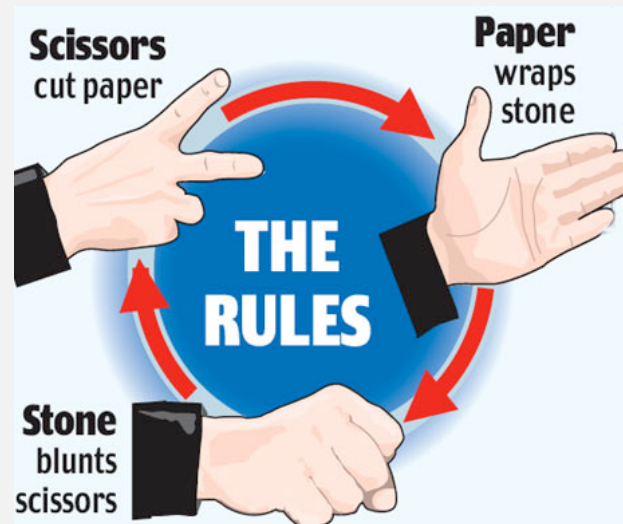
## Total order

A **total order** is a binary relation  $\leq$  that satisfies

- **Antisymmetry:** if  $v \leq w$  and  $w \leq v$ , then  $v = w$ .
- **Transitivity:** if  $v \leq w$  and  $w \leq x$ , then  $v \leq x$ .
- **Totality:** either  $v \leq w$  or  $w \leq v$  or both.

Ex.

- Standard order for natural numbers and real numbers.
- Alphabetical order for strings.
- Chronological order for dates.
- ...

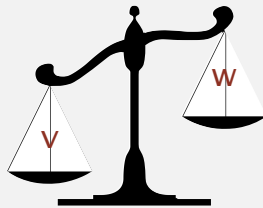


an intransitive relation

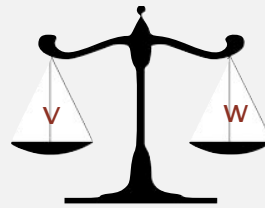
## Comparable API

Implement `compareTo()` so that `v.compareTo(w)`

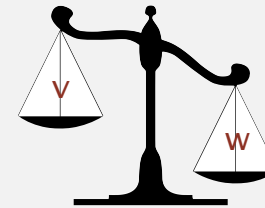
- Implements a total order.
- Returns a negative integer, zero, or positive integer if  $v$  is less than, equal to, or greater than  $w$ , respectively.
- Throws an exception if incompatible types (or either is `null`).



less than (return -1)



equal to (return 0)



greater than (return +1)

**Built-in comparable types.** Integer, Double, String, Date, File, ...

**User-defined comparable types.** Implement the `Comparable` interface.

## Implementing the Comparable interface

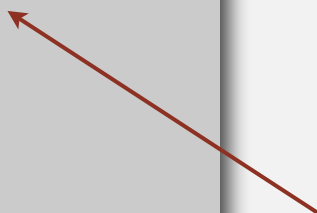
Date data type. Simplified version of `java.util.Date`.

```
public class Date implements Comparable<Date>
{
    private final int month, day, year;

    public Date(int m, int d, int y)
    {
        month = m;
        day    = d;
        year   = y;
    }

    public int compareTo(Date that)
    {
        if (this.year < that.year ) return -1;
        if (this.year > that.year ) return +1;
        if (this.month < that.month) return -1;
        if (this.month > that.month) return +1;
        if (this.day < that.day ) return -1;
        if (this.day > that.day ) return +1;
        return 0;
    }
}
```

only compare dates  
to other dates



## Two useful sorting abstractions

**Helper functions.** Refer to data through compares and exchanges.

**Less.** Is item  $v$  less than  $w$ ?

```
private static boolean less(Comparable v, Comparable w)
{ return v.compareTo(w) < 0; }
```

**Exchange.** Swap item in array  $a[]$  at index  $i$  with the one at index  $j$ .

```
private static void exch(Comparable[] a, int i, int j)
{
    Comparable swap = a[i];
    a[i] = a[j];
    a[j] = swap;
}
```

## Testing

Goal. Test if an array is sorted.

```
private static boolean isSorted(Comparable[] a)
{
    for (int i = 1; i < a.length; i++)
        if (less(a[i], a[i-1])) return false;
    return true;
}
```

Q. If the sorting algorithm passes the test, did it correctly sort the array?

A.

- ▶ rules of the game
- ▶ **selection sort**
- ▶ insertion sort
- ▶ shellsort
- ▶ shuffling
- ▶ convex hull

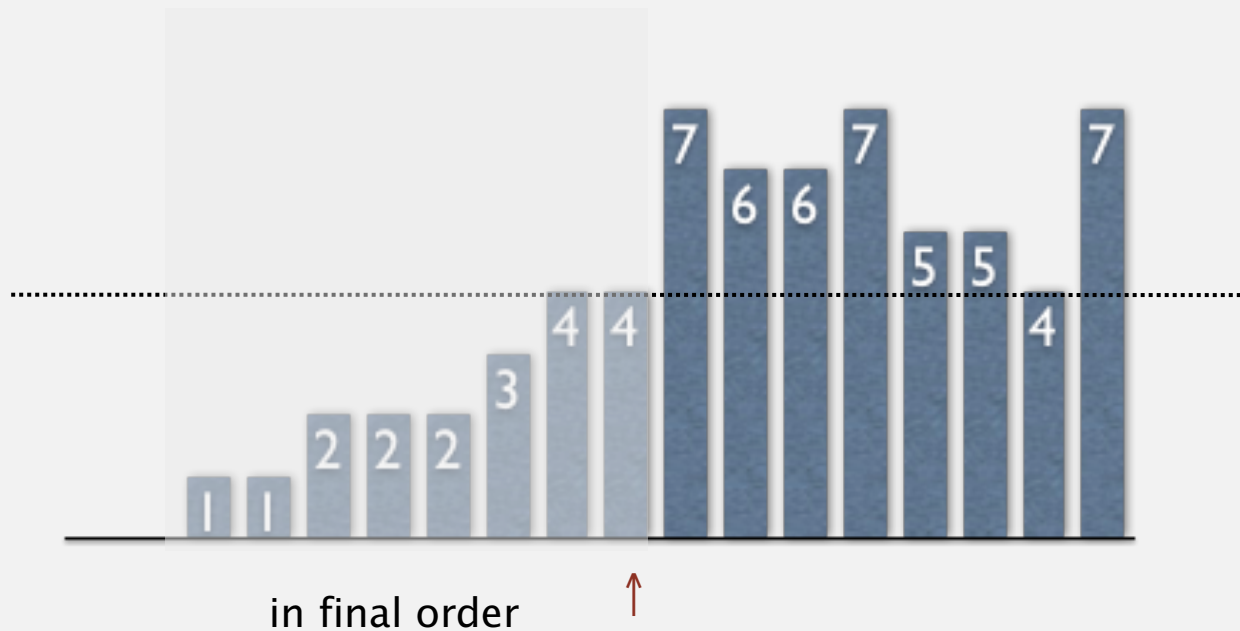
## Selection sort demo

## Selection sort

Algorithm. ↑ scans from left to right.

### Invariants.

- Entries the left of ↑ (including ↑) fixed and in ascending order.
- No entry to right of ↑ is smaller than any entry to the left of ↑.





## Selection sort inner loop

To maintain algorithm invariants:

- Move the pointer to the right.

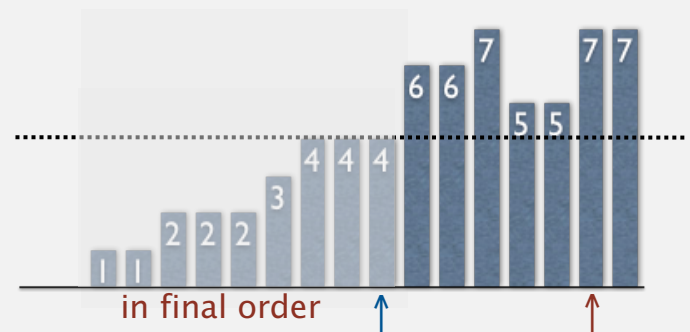
```
i++;
```

- Identify index of minimum entry on right.

```
int min = i;  
for (int j = i+1; j < N; j++)  
    if (less(a[j], a[min]))  
        min = j;
```

- Exchange into position.

```
exch(a, i, min);
```



## Selection sort: Java implementation

```
public class Selection
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
        {
            int min = i;
            for (int j = i+1; j < N; j++)
                if (less(a[j], a[min]))
                    min = j;
            exch(a, i, min);
        }
    }

    private static boolean less(Comparable v, Comparable w)
    { /* as before */ }

    private static void exch(Comparable[] a, int i, int j)
    { /* as before */ }
}
```

## Selection sort: mathematical analysis

**Proposition.** Selection sort uses  $(N-1) + (N-2) + \dots + 1 + 0 \sim N^2/2$  compares and  $N$  exchanges.

		a[]										
i	min	0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
0	6	S	O	R	T	E	X	A	M	P	L	E
1	4	A	O	R	T	E	X	S	M	P	L	E
2	10	A	E	R	T	O	X	S	M	P	L	E
3	9	A	E	E	T	O	X	S	M	P	L	R
4	7	A	E	E	L	O	X	S	M	P	T	R
5	7	A	E	E	L	M	X	S	O	P	T	R
6	8	A	E	E	L	M	O	S	X	P	T	R
7	10	A	E	E	L	M	O	P	X	S	T	R
8	8	A	E	E	L	M	O	P	R	S	T	X
9	9	A	E	E	L	M	O	P	R	S	T	X
10	10	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

Trace of selection sort (array contents just after each exchange)

*entries in black are examined to find the minimum*

*entries in red are a[min]*

*entries in gray are in final position*

Running time insensitive to input. Quadratic time, even if input array is sorted.  
Data movement is minimal. Linear number of exchanges.

## Selection sort: animations

20 random items

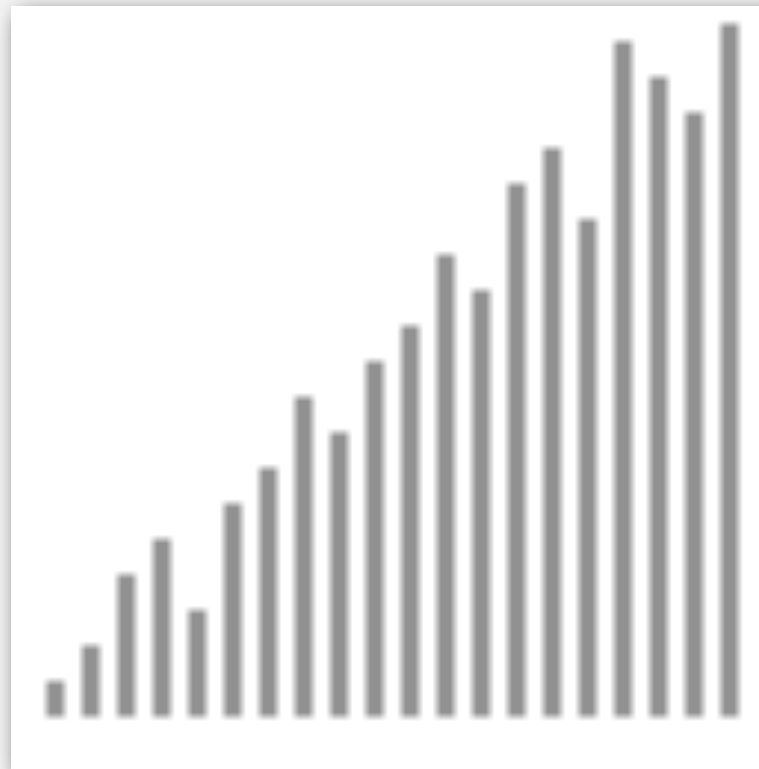


- ▲ algorithm position
- █ in final order
- ▒ not in final order

<http://www.sorting-algorithms.com/selection-sort>

## Selection sort: animations

20 partially-sorted items



- ▲ algorithm position
- █ in final order
- ▒ not in final order

<http://www.sorting-algorithms.com/selection-sort>

## Selection sort: *Gypsy folk dance*



- ▶ rules of the game
- ▶ selection sort
- ▶ **insertion sort**
- ▶ shellsort
- ▶ shuffling
- ▶ convex hull

## Insertion sort demo



## Insertion sort

Algorithm. ↑ scans from left to right.

### Invariants.

- Entries to the left of ↑ (including ↑) are in ascending order.
- Entries to the right of ↑ have not yet been seen.



## Insertion sort inner loop

To maintain algorithm invariants:

- Move the pointer to the right.

```
i++;
```



- Moving from right to left, exchange  $a[i]$  with each larger entry to its left.

```
for (int j = i; j > 0; j--)  
    if (less(a[j], a[j-1]))  
        exch(a, j, j-1);  
    else break;
```



## Insertion sort: Java implementation

```
public class Insertion
{
    public static void sort(Comparable[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
            for (int j = i; j > 0; j--)
                if (less(a[j], a[j-1]))
                    exch(a, j, j-1);
                else break;
    }

    private static boolean less(Comparable v, Comparable w)
    { /* as before */ }

    private static void exch(Comparable[] a, int i, int j)
    { /* as before */ }
}
```

## Insertion sort: mathematical analysis

**Proposition.** To sort a randomly-ordered array with distinct keys, insertion sort uses  $\sim \frac{1}{4} N^2$  compares and  $\sim \frac{1}{4} N^2$  exchanges on average.

**Pf.** Expect each entry to move halfway back.

		a[]										
i	j	0	1	2	3	4	5	6	7	8	9	10
		S	O	R	T	E	X	A	M	P	L	E
1	0	O	S	R	T	E	X	A	M	P	L	E
2	1	O	R	S	T	E	X	A	M	P	L	E
3	3	O	R	S	T	E	X	A	M	P	L	E
4	0	E	O	R	S	T	X	A	M	P	L	E
5	5	E	O	R	S	T	X	A	M	P	L	E
6	0	A	E	O	R	S	T	X	M	P	L	E
7	2	A	E	M	O	R	S	T	X	P	L	E
8	4	A	E	M	O	P	R	S	T	X	L	E
9	2	A	E	L	M	O	P	R	S	T	X	E
10	2	A	E	E	L	M	O	P	R	S	T	X
		A	E	E	L	M	O	P	R	S	T	X

Trace of insertion sort (array contents just after each insertion)

*entries in gray do not move*

*entry in red is a[j]*

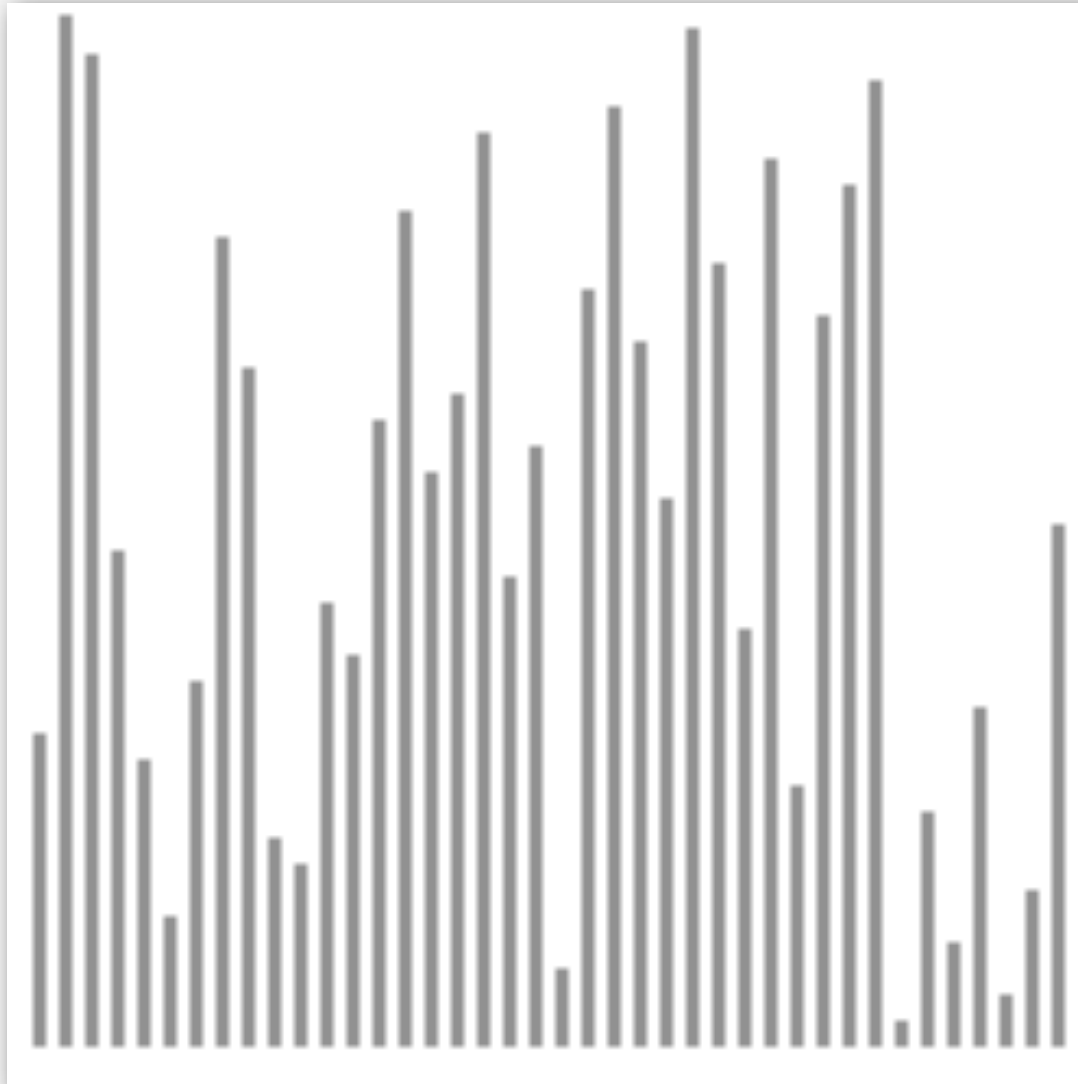
*entries in black moved one position right for insertion*

# Insertion sort: trace

		a[]																																				
i	j	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34		
		A	S	O	M	E	W	H	A	T	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
0	0	A	S	O	M	E	W	H	A	T	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
1	1	A	S	O	M	E	W	H	A	T	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
2	1	A	O	S	M	E	W	H	A	T	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
3	1	A	M	O	S	E	W	H	A	T	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
4	1	A	E	M	O	S	W	H	A	T	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
5	5	A	E	M	O	S	W	H	A	T	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
6	2	A	E	H	M	O	S	W	A	T	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
7	1	A	A	E	H	M	O	S	W	T	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
8	7	A	A	E	H	M	O	S	T	W	L	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
9	4	A	A	E	H	L	M	O	S	T	W	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
10	7	A	A	E	H	L	M	O	S	T	W	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
11	6	A	A	E	H	L	M	N	O	O	S	T	W	O	N	G	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E
12	3	A	A	E	G	H	L	M	N	O	O	S	T	W	E	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
13	3	A	A	E	E	G	H	L	M	N	O	O	S	T	W	R	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
14	11	A	A	E	E	G	H	L	M	N	O	O	R	S	T	W	I	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
15	6	A	A	E	E	G	H	I	L	M	N	O	O	R	S	T	W	N	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
16	10	A	A	E	E	G	H	I	L	M	N	N	O	O	R	S	T	W	S	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
17	15	A	A	E	E	G	H	I	L	M	N	N	O	O	R	S	S	T	W	E	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
18	4	A	A	E	E	E	G	H	I	L	M	N	N	O	O	R	S	S	T	W	R	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
19	15	A	A	E	E	E	G	H	I	L	M	N	N	O	O	R	R	S	S	T	W	T	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
20	19	A	A	E	E	E	G	H	I	L	M	N	N	O	O	R	R	S	S	T	T	W	I	O	N	S	O	R	T	E	X	A	M	P	L	E		
21	8	A	A	E	E	E	G	H	I	I	L	M	N	N	O	O	R	R	S	S	T	T	W	O	N	S	O	R	T	E	X	A	M	P	L	E		
22	15	A	A	E	E	E	G	H	I	I	L	M	N	N	O	O	O	R	R	S	S	T	T	W	N	S	O	R	T	E	X	A	M	P	L	E		
23	13	A	A	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	R	R	S	S	T	T	W	S	O	R	T	E	X	A	M	P	L	E		
24	21	A	A	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	R	R	S	S	S	T	T	W	O	R	T	E	X	A	M	P	L	E		
25	17	A	A	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	S	S	S	T	T	W	R	T	E	X	A	M	P	L	E		
26	20	A	A	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	W	T	E	X	A	M	P	L	E		
27	26	A	A	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	E	X	A	M	P	L	E		
28	5	A	A	E	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	X	A	M	P	L	E		
29	29	A	A	E	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	X	A	M	P	L	E		
30	2	A	A	A	E	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	X	M	P	L	E		
31	13	A	A	A	E	E	E	E	G	H	I	I	L	M	N	N	N	O	O	O	O	R	R	R	S	S	S	T	T	T	W	X	P	L	E			
32	21	A	A	A	E	E	E	E	G	H	I	I	L	M	M	N	N	N	O	O	O	O	P	R	R	R	S	S	S	T	T	T	W	X	L	E		
33	12	A	A	A	E	E	E	E	G	H	I	I	L	L	M	M	N	N	N	O	O	O	O	P	R	R	R	S	S	S	T	T	T	W	X	E		
34	7	A	A	A	E	E	E	E	E	G	H	I	I	L	L	M	M	N	N	N	O	O	O	O	P	R	R	R	S	S	S	T	T	T	W	X		
		A	A	A	E	E	E	E	E	G	H	I	I	L	L	M	M	N	N	N	O	O	O	O	P	R	R	R	S	S	S	T	T	T	W	X		

# Insertion sort: animation

40 random items



- ▲ algorithm position
- █ in order
- ▒ not yet seen

<http://www.sorting-algorithms.com/insertion-sort>

## Insertion sort: best and worst case

**Best case.** If the array is in ascending order, insertion sort makes  $N-1$  compares and 0 exchanges.

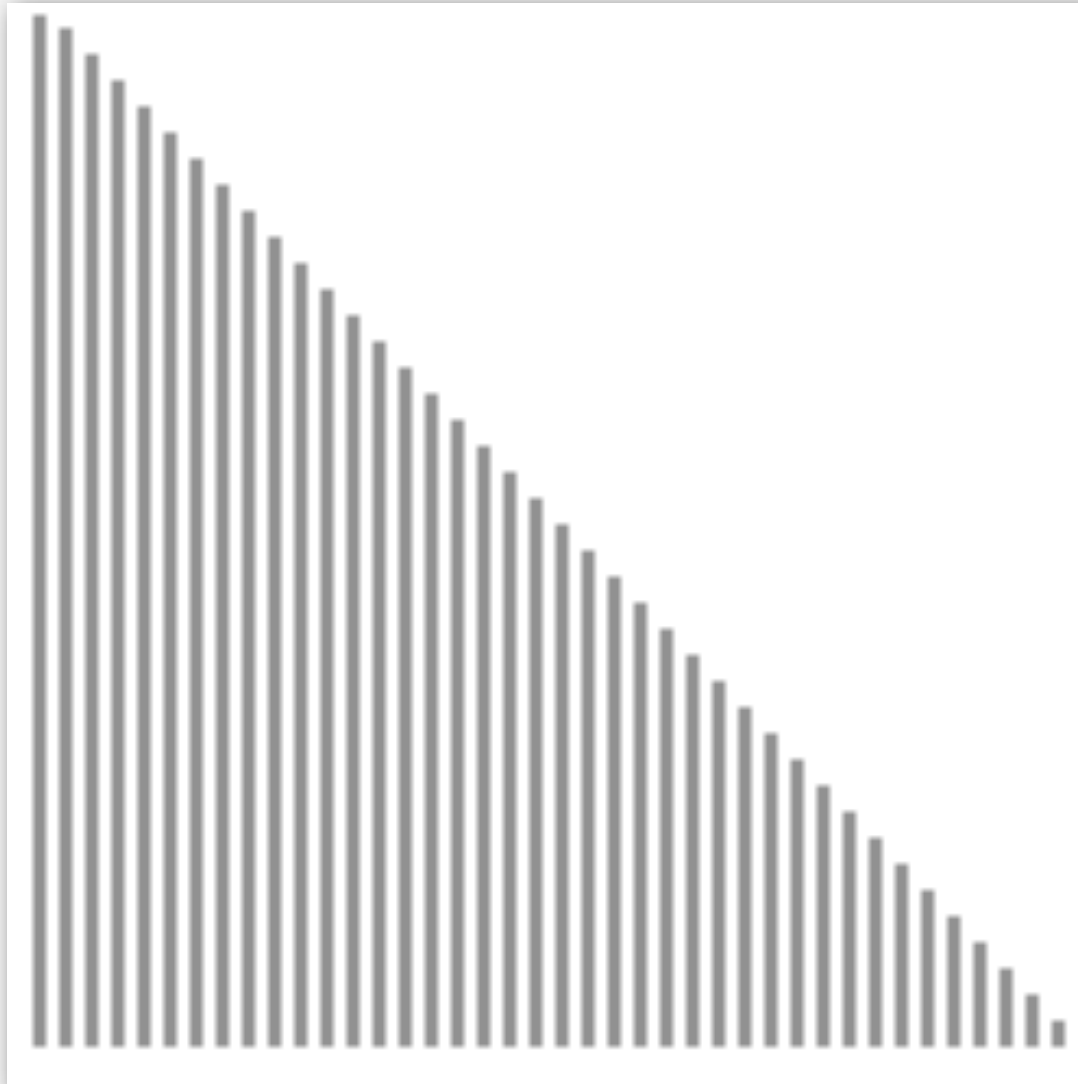
A E E L M O P R S T X

**Worst case.** If the array is in descending order (and no duplicates), insertion sort makes  $\sim \frac{1}{2} N^2$  compares and  $\sim \frac{1}{2} N^2$  exchanges.

X T S R P O M L E E A

# Insertion sort: animation

40 reverse-sorted items



<http://www.sorting-algorithms.com/insertion-sort>

- ▲ algorithm position
- ▬ in order
- ▬ not yet seen



## Insertion sort: partially-sorted arrays

Def. An **inversion** is a pair of keys that are out of order.

A E E L M O T R X P S

T-R T-P T-S R-P X-P X-S

(6 inversions)

Def. An array is **partially sorted** if the number of inversions is  $\leq cN$ .

- Ex 1. A subarray of size 10 appended to a sorted subarray of size  $N$ .
- Ex 2. An array of size  $N$  with only 10 entries out of place.

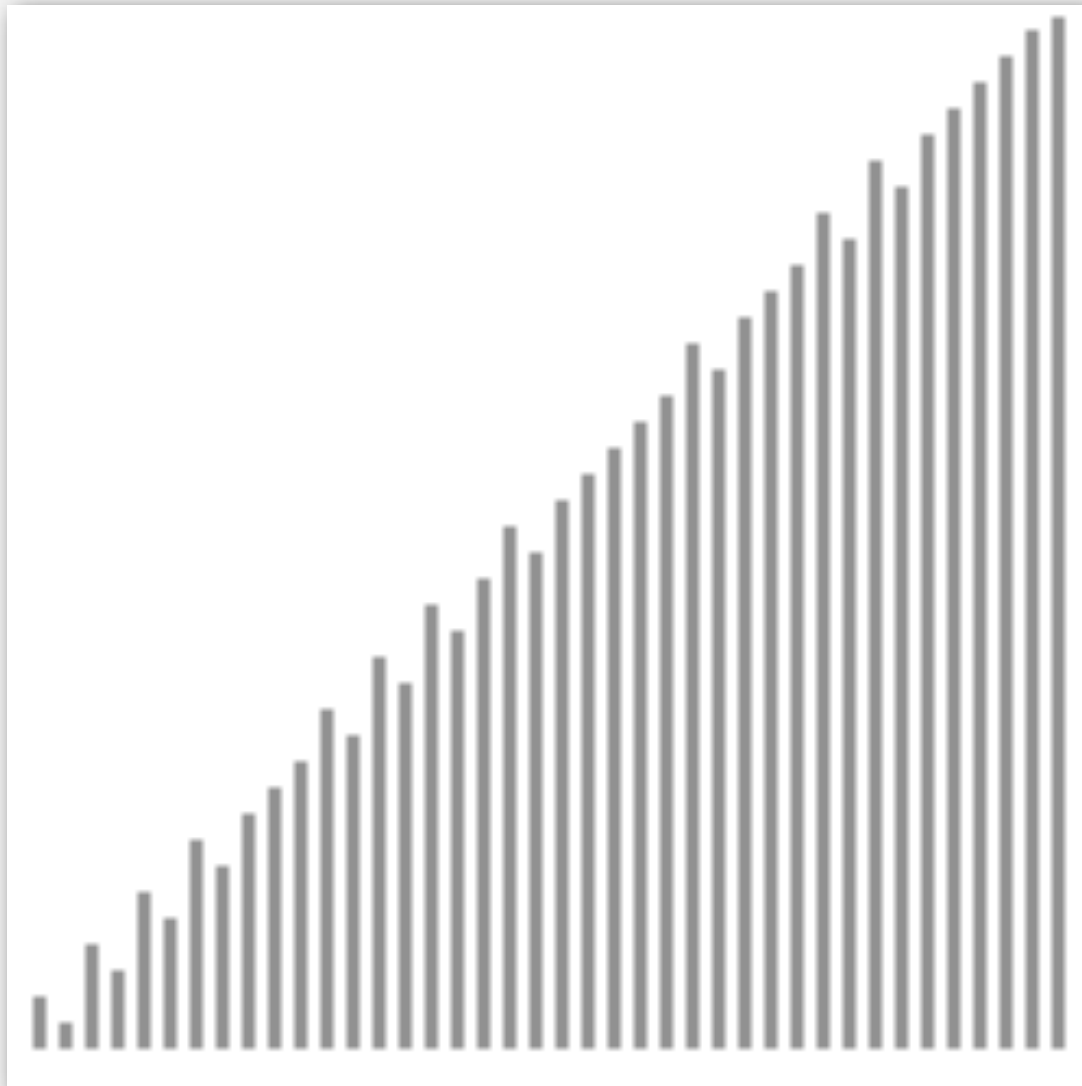
Proposition. For partially-sorted arrays, insertion sort runs in linear time.

Pf. Number of exchanges equals the number of inversions.

↑  
number of compares = exchanges +  $(N - 1)$

# Insertion sort: animation

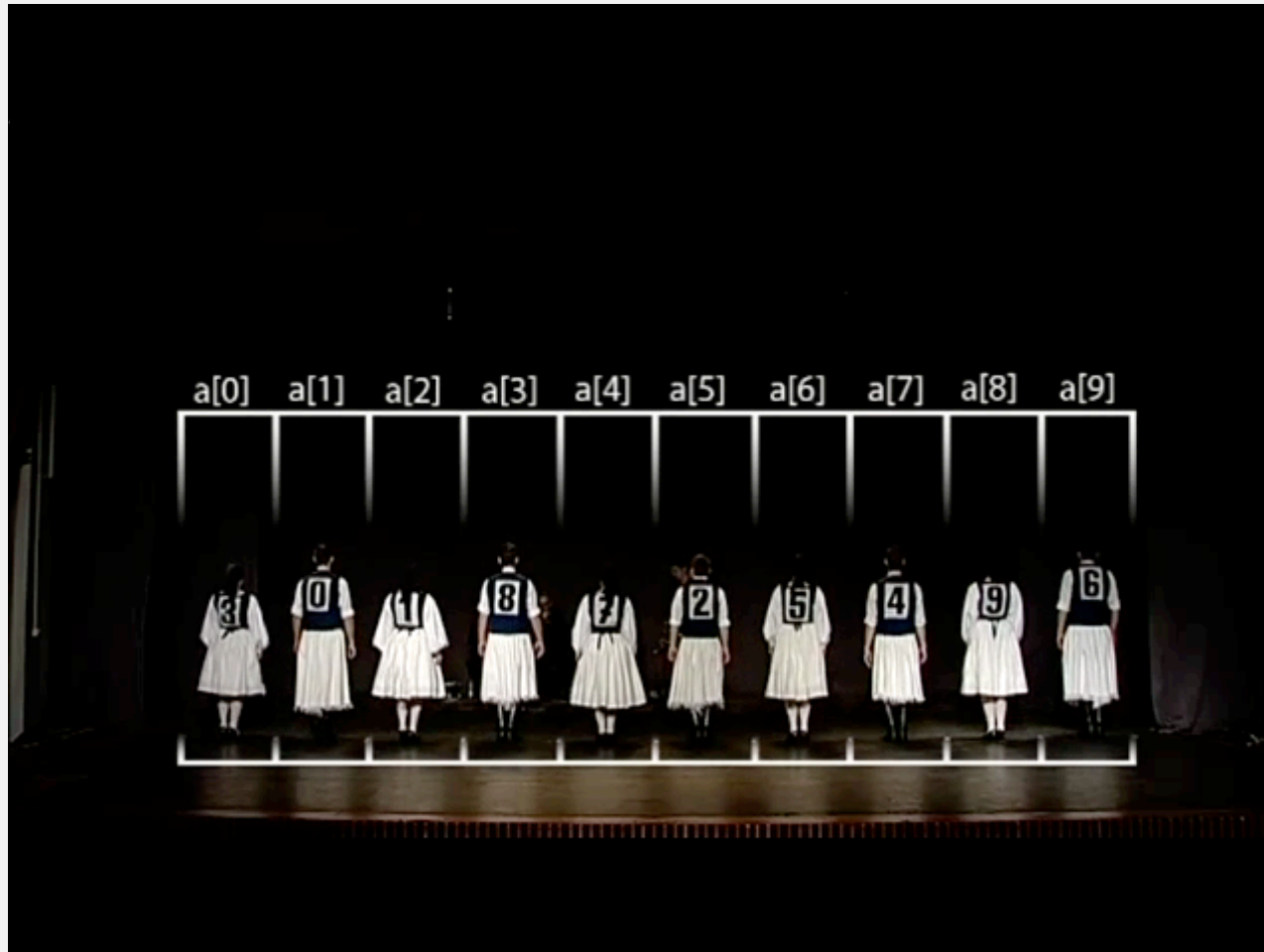
40 partially-sorted items



<http://www.sorting-algorithms.com/insertion-sort>

- ▲ algorithm position
- █ in order
- █ not yet seen

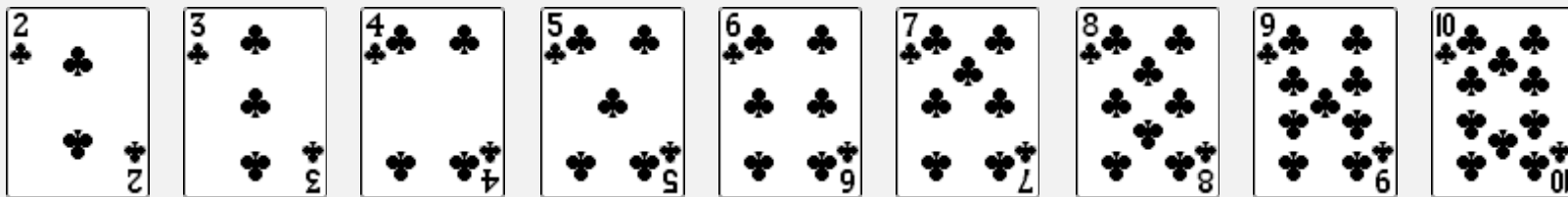
## Insertion sort: Romanian folk dance



- ▶ rules of the game
- ▶ selection sort
- ▶ insertion sort
- ▶ shellsort
- ▶ **shuffling**
- ▶ convex hull

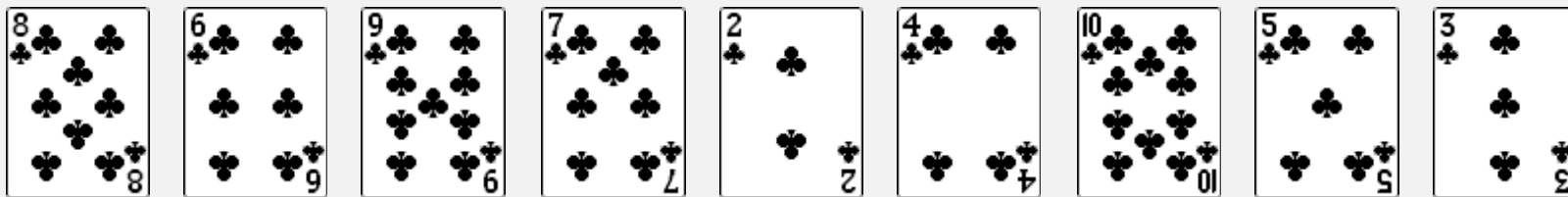
## How to shuffle an array

**Shuffling.** Rearrange an array so that result is a uniformly random permutation.



## How to shuffle an array

**Shuffling.** Rearrange an array so that result is a uniformly random permutation.

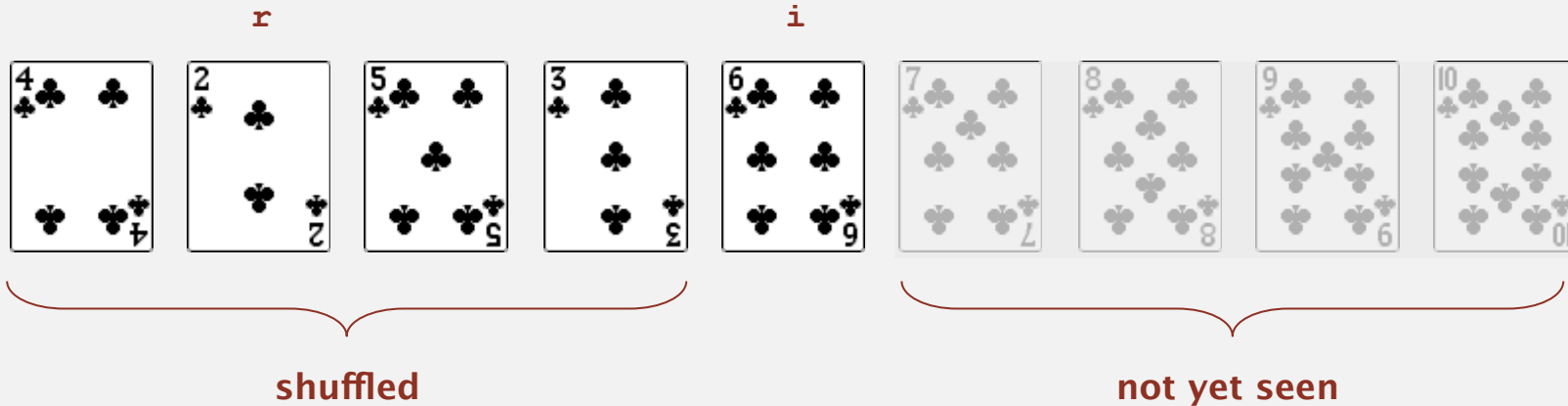


## Knuth shuffle demo

# Knuth shuffle

Knuth shuffle. [Fisher-Yates 1938]

- In iteration  $i$ , pick integer  $r$  between 0 and  $i$  uniformly at random.
- Swap  $a[i]$  and  $a[r]$ .



**Proposition.** Knuth shuffling algorithm produces a uniformly random permutation of the input array in linear time.

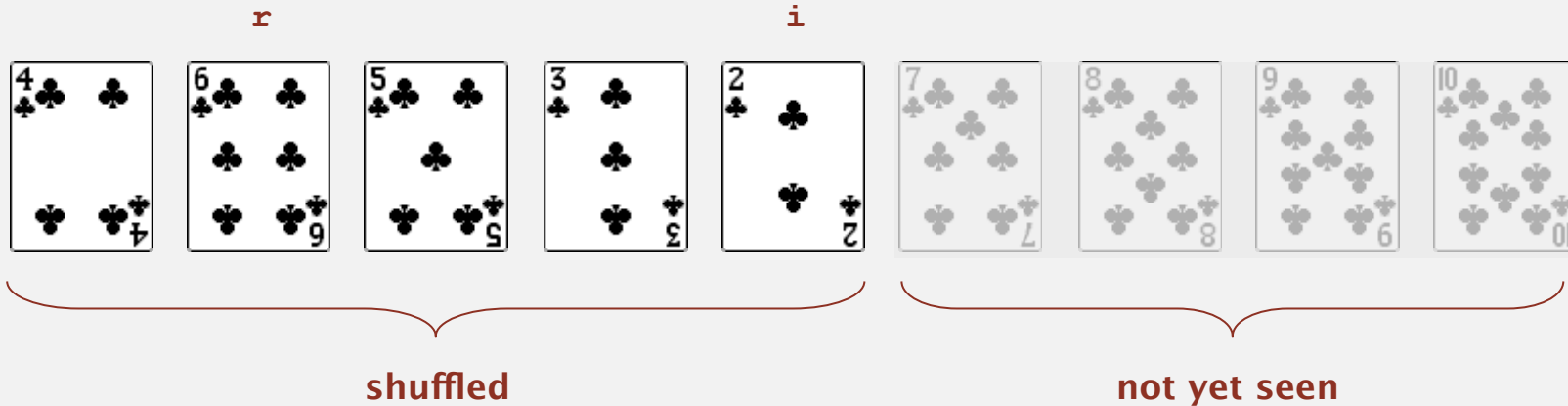
↖ assuming integers uniformly at random



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- Swap  $a[i]$  and  $a[r]$ .

common bug: between 0 and  $N - 1$   
correct variant: between  $i$  and  $N - 1$

```
public class StdRandom
{
    ...
    public static void shuffle(Object[] a)
    {
        int N = a.length;
        for (int i = 0; i < N; i++)
        {
            int r = StdRandom.uniform(i + 1);
            exch(a, i, r);
        }
    }
}
```

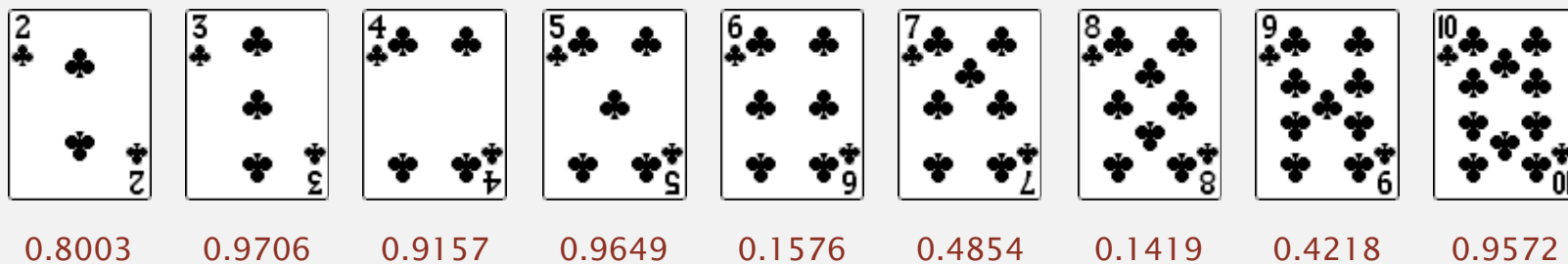
← between 0 and  $i$

# Shuffle sort

## Shuffle sort.

- Generate a random real number for each array entry.
- Sort the array.

↑ useful for shuffling  
columns in a spreadsheet



**Proposition.** Shuffle sort produces a uniformly random permutation of the input array, provided no duplicate values.

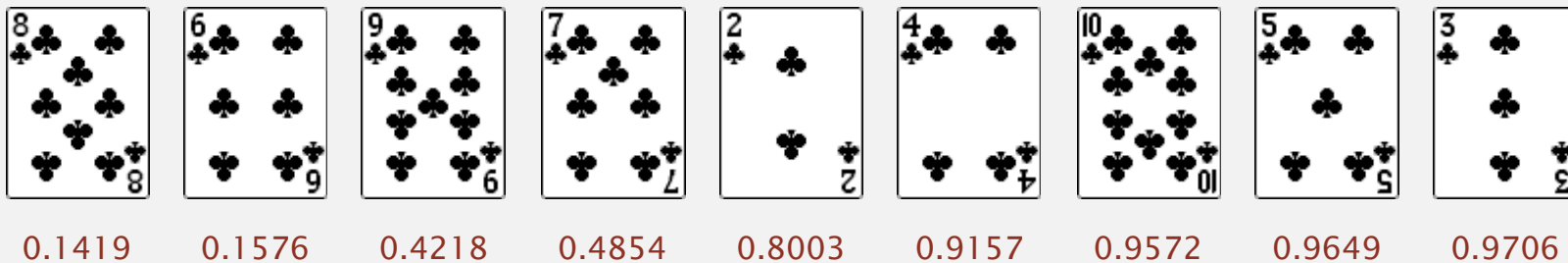
↑ assuming real numbers  
uniformly at random

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## War story (Microsoft)

Microsoft antitrust probe by EU. Microsoft agreed to provide a randomized ballot screen for users to select browser in Windows 7.

<http://www.browserchoice.eu>

### Select your web browser(s)



A fast new browser from Google. Try it now!



Safari for Windows from Apple, the world's most innovative browser.



Your online security is Firefox's top priority. Firefox is free, and made to help you get the most out of the



The fastest browser on Earth. Secure, powerful and easy to use, with excellent privacy protection.



Designed to help you take control of your privacy and browse with confidence. Free from Microsoft.



appeared last 50% of the time

## War story (Microsoft)

**Microsoft antitrust probe by EU.** Microsoft agreed to provide a randomized ballot screen for users to select browser in Windows 7.

**Solution?** Implement shuffle sort by making comparator always return a random answer.

```
public int compareTo(Browser that)
{
    double r = Math.random();
    if (r < 0.5) return -1;
    if (r > 0.5) return +1;
    return 0;
}
```

← browser comparator  
(should implement a total order)

## War story (online poker)

Texas hold'em poker. Software must shuffle electronic cards.



How We Learned to Cheat at Online Poker: A Study in Software Security  
<http://itmanagement.earthweb.com/entdev/article.php/616221>

## War story (online poker)

Shuffling algorithm in FAQ at [www.planetpoker.com](http://www.planetpoker.com)

```
for i := 1 to 52 do begin
  r := random(51) + 1;
  swap := card[r];
  card[r] := card[i];
  card[i] := swap;
end;
```

← between 1 and 51

- Bug 1. Random number  $r$  never 52  $\Rightarrow$  52<sup>nd</sup> card can't end up in 52<sup>nd</sup> place.
- Bug 2. Shuffle not uniform (should be between  $i$  and 51).
- Bug 3. `random()` uses 32-bit seed  $\Rightarrow$   $2^{32}$  possible shuffles.
- Bug 4. Seed = milliseconds since midnight  $\Rightarrow$  86.4 million possible shuffles.

*“ The generation of random numbers is too important to be left to chance. ”*

*— Robert R. Coveyou*



## War story (online poker)

### Best practices for shuffling (if your business depends on it).

- Use a hardware random-number generator that has passed both the FIPS 140-2 and the NIST statistical test suites.
- Continuously monitor statistic properties:  
hardware random-number generators are fragile and fail silently.
- Use an unbiased shuffling algorithm.

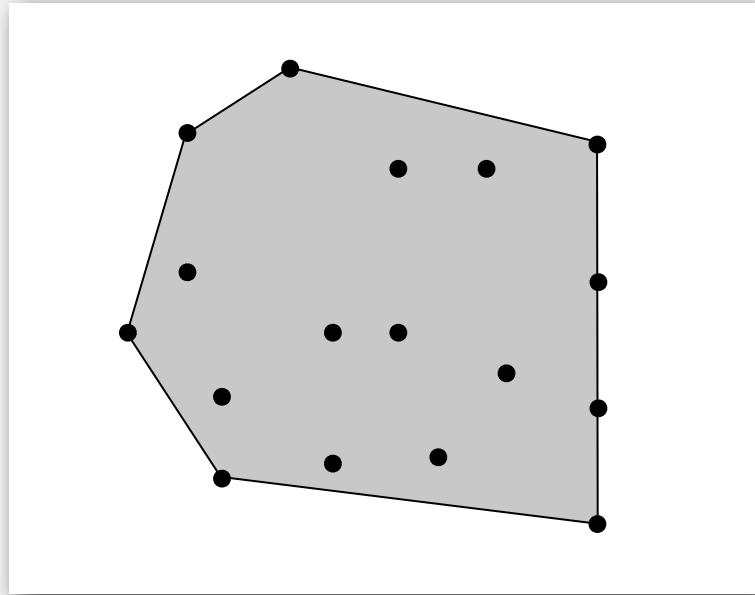


**Bottom line.** Shuffling a deck of cards is hard!

- ▶ rules of the game
- ▶ selection sort
- ▶ insertion sort
- ▶ shellsort
- ▶ shuffling
- ▶ **convex hull**

## Convex hull

The **convex hull** of a set of  $N$  points is the smallest perimeter fence enclosing the points.

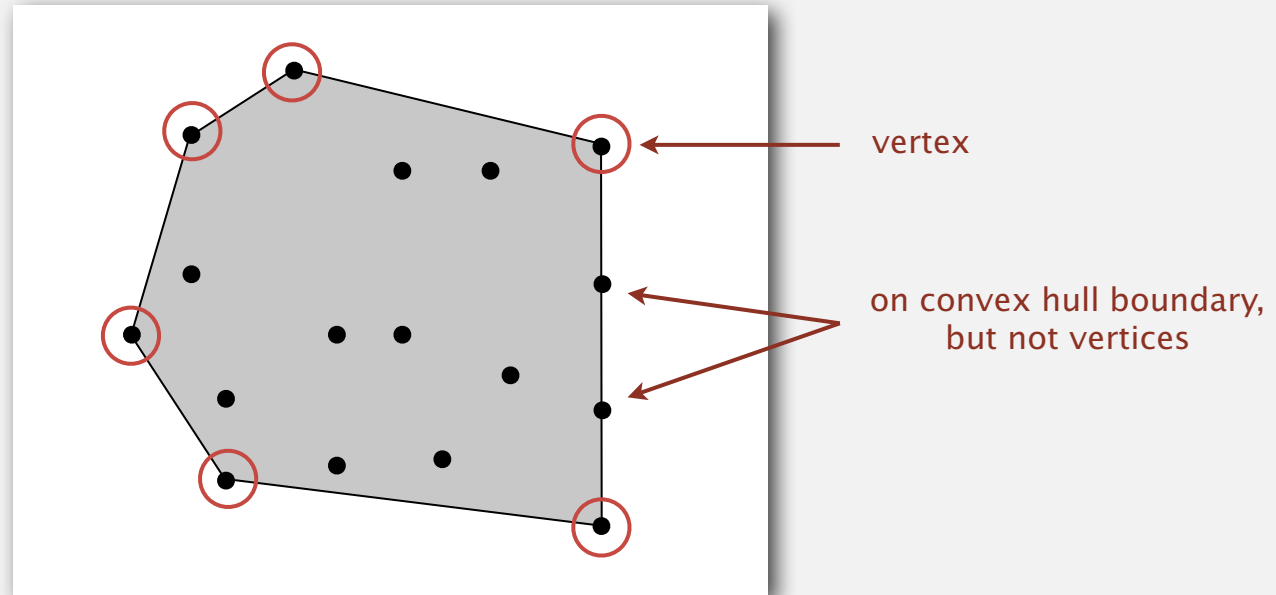


### Equivalent definitions.

- Smallest convex set containing all the points.
- Smallest area convex polygon enclosing the points.
- Convex polygon enclosing the points, whose vertices are points in the set.

## Convex hull

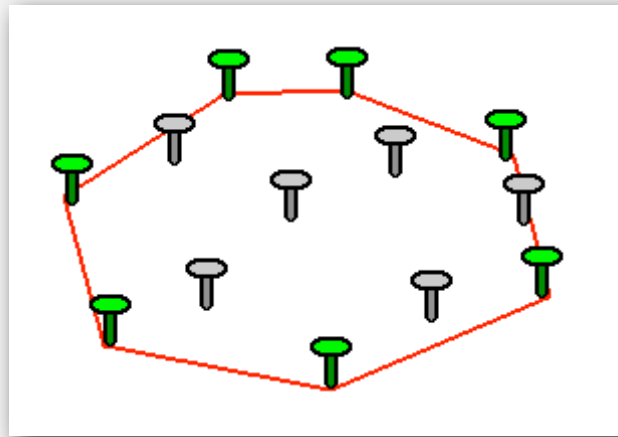
The **convex hull** of a set of  $N$  points is the smallest perimeter fence enclosing the points.



**Convex hull output.** Sequence of vertices in counterclockwise order.

## Convex hull: mechanical algorithm

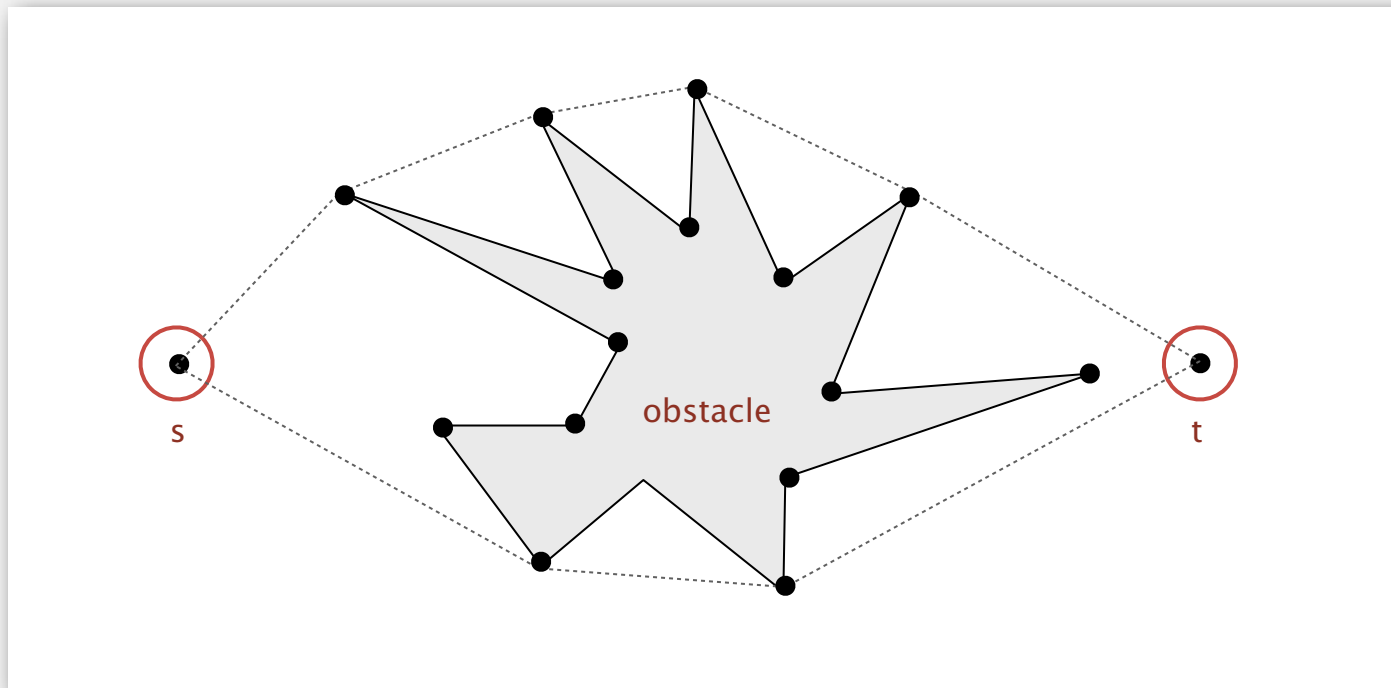
**Mechanical algorithm.** Hammer nails perpendicular to plane; stretch elastic rubber band around points.



[http://www.dfanning.com/math\\_tips/convexhull\\_1.gif](http://www.dfanning.com/math_tips/convexhull_1.gif)

## Convex hull application: motion planning

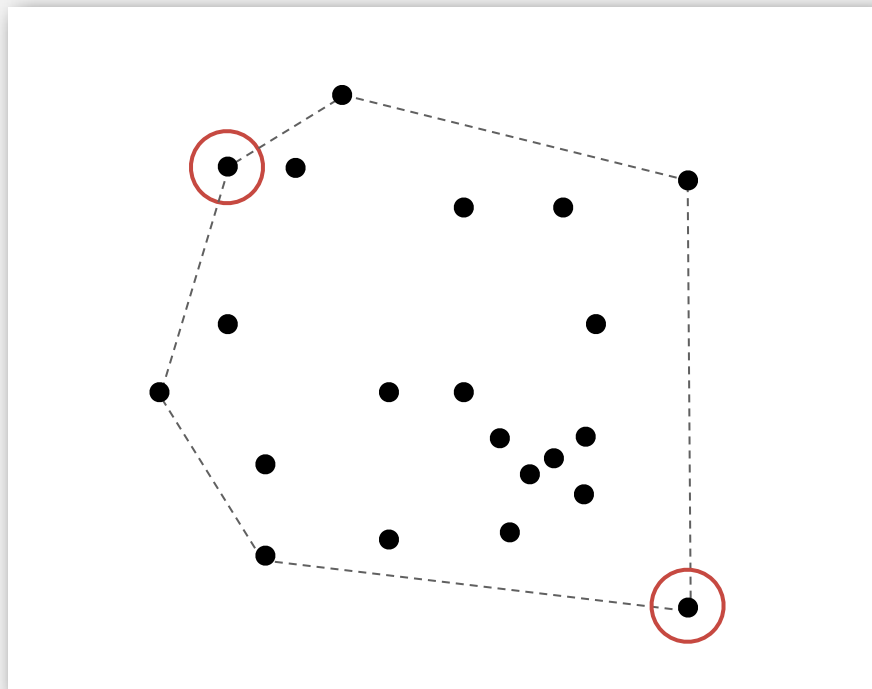
**Robot motion planning.** Find shortest path in the plane from  $s$  to  $t$  that avoids a polygonal obstacle.



**Fact.** Shortest path is either straight line from  $s$  to  $t$  or it is one of two polygonal chains of convex hull.

## Convex hull application: farthest pair

**Farthest pair problem.** Given  $N$  points in the plane, find a pair of points with the largest Euclidean distance between them.

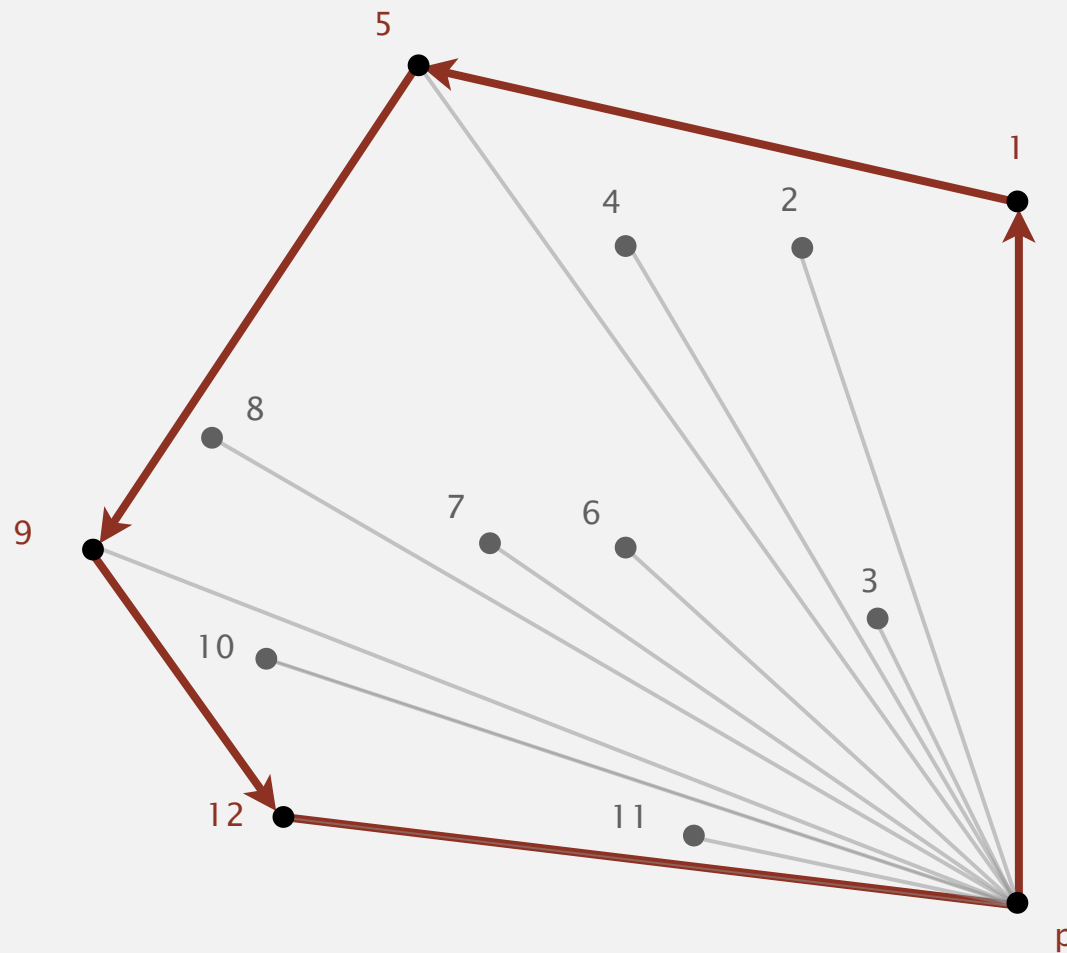


**Fact.** Farthest pair of points are extreme points on convex hull.

## Convex hull: geometric properties

**Fact.** Can traverse the convex hull by making only counterclockwise turns.

**Fact.** The vertices of convex hull appear in increasing order of polar angle with respect to point  $p$  with lowest  $y$ -coordinate.

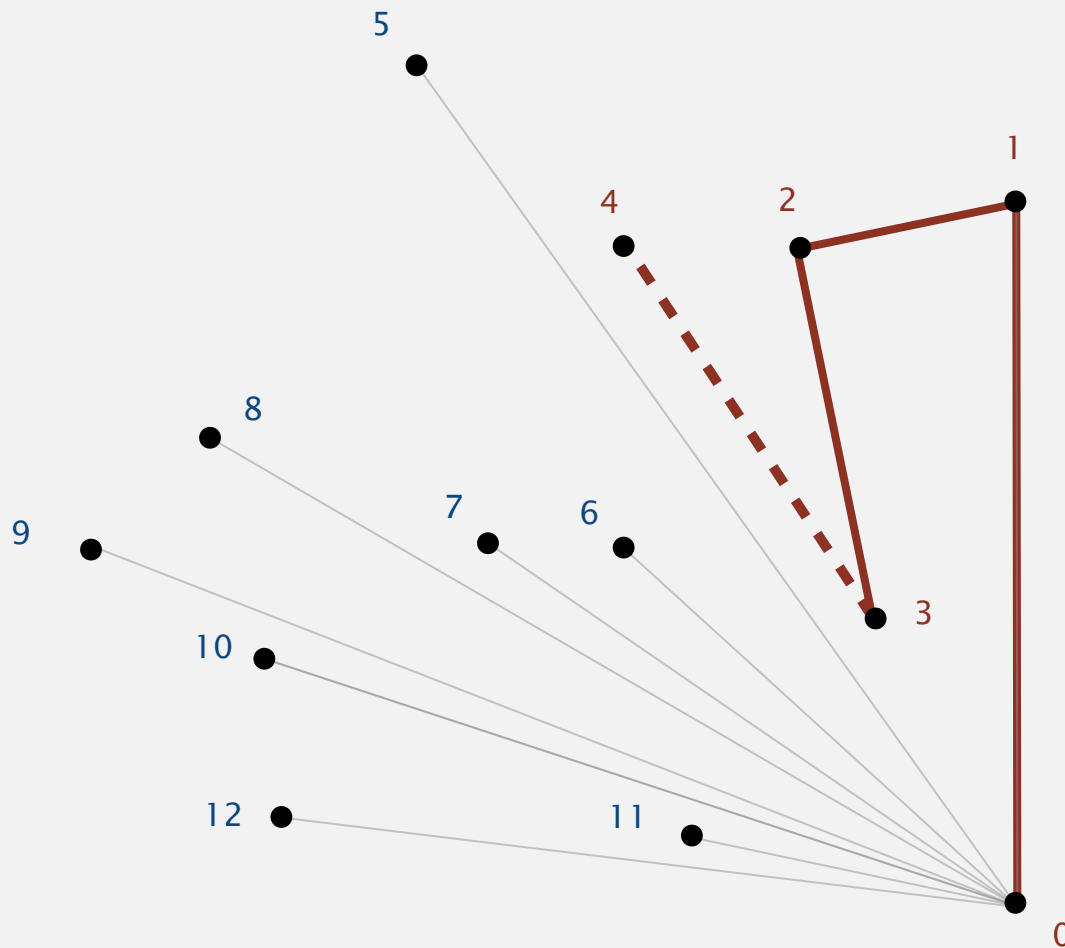




## Graham scan demo

## Convex hull: Graham scan

- Choose point  $p$  with smallest  $y$ -coordinate.
- Sort points by polar angle with  $p$ .
- Consider points in order, and discard unless that would create a ccw turn.



## Graham scan: implementation challenges

Q. How to find point  $p$  with smallest  $y$ -coordinate?

A. Define a total order, comparing  $y$ -coordinate. [next lecture]

Q. How to sort points by polar angle with respect to  $p$  ?

A. Define a total order **for each** point  $p$ . [next lecture]

Q. How to determine whether  $p_1 \rightarrow p_2 \rightarrow p_3$  is a counterclockwise turn?

A. Computational geometry. [next two slides]

Q. How to sort efficiently?

A. Mergesort sorts in  $N \log N$  time. [next lecture]

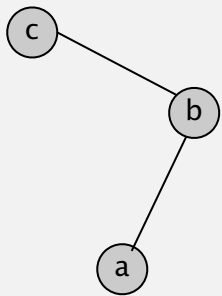
Q. How to handle degeneracies (three or more points on a line)?

A. Requires some care, but not hard. [see booksite]

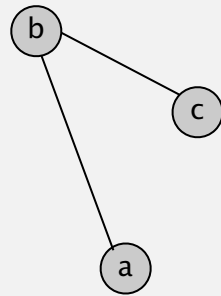
## Implementing ccw

**CCW.** Given three points  $a$ ,  $b$ , and  $c$ , is  $a \rightarrow b \rightarrow c$  a counterclockwise turn?

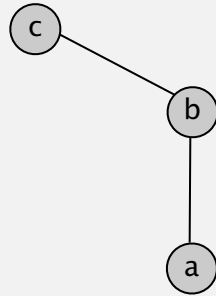
is  $c$  to the left of the ray  $a \rightarrow b$



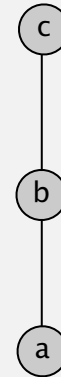
yes



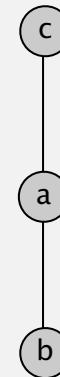
no



yes  
( $\infty$ -slope)



no  
(collinear)



no  
(collinear)



no  
(collinear)

**Lesson.** Geometric primitives are tricky to implement.

- Dealing with degenerate cases.
- Coping with floating-point precision.

## Implementing ccw

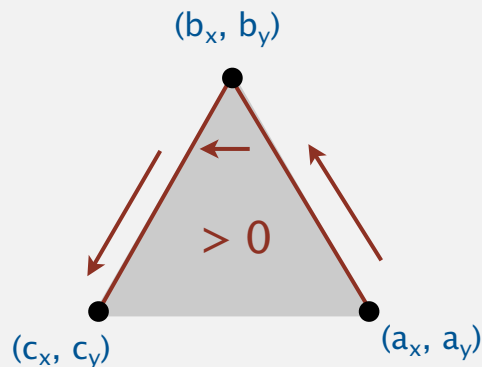
**CCW.** Given three points  $a$ ,  $b$ , and  $c$ , is  $a \rightarrow b \rightarrow c$  a counterclockwise turn?

- Determinant (or cross product) gives twice signed area of planar triangle.

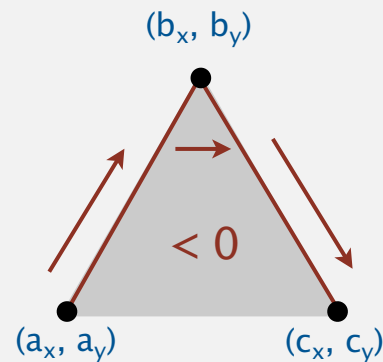
$$2 \times \text{Area}(a, b, c) = \begin{vmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{vmatrix} = (b_x - a_x)(c_y - a_y) - (b_y - a_y)(c_x - a_x)$$

$(b - a) \times (c - a)$

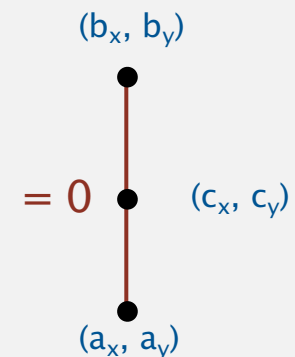
- If signed area  $> 0$ , then  $a \rightarrow b \rightarrow c$  is counterclockwise.
- If signed area  $< 0$ , then  $a \rightarrow b \rightarrow c$  is clockwise.
- If signed area  $= 0$ , then  $a \rightarrow b \rightarrow c$  are collinear.



counterclockwise



clockwise



collinear

## Immutable point data type

```
public class Point2D
{
    private final double x;
    private final double y;

    public Point(double x, double y)
    {
        this.x = x;
        this.y = y;
    }

    ...

    public static int ccw(Point a, Point b, Point c)
    {
        int area2 = (b.x-a.x)*(c.y-a.y) - (b.y-a.y)*(c.x-a.x);
        if (area2 < 0) return -1; // clockwise
        else if (area2 > 0) return +1; // counter-clockwise
        else return 0; // collinear
    }
}
```

danger of floating-point roundoff error

